Thermodynamically-consistent mesoscopic model of the ferro/paramagnetic transition

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Abstract: A continuum evolutionary model for micromagnetics is presented that, beside the standard magnetic balance laws, includes thermo-magnetic coupling. To allow conceptually efficient computer implementation, inspired by relaxation method of static minimization problems, our model is mesoscopic in the sense that possible fine spatial oscillations of the magnetization are modeled by means of Young measures. Existence of weak solutions is proved by backward Euler time discretization.

AMS Subj. Classification: 35K85, 35Q60, 49S05, 78A30, 78M30, 80A17.

Key words: micromagnetism, relaxation, Young measures, thermodynamics.

1. Introduction, static problem and its relaxation

Micromagnetics is a continuum theory introduced by Brown [5] to describe the equilibrium states of saturated ferromagnets. The equilibria are determined as minimizers of a functional with exchange, anisotropy, Zeeman (interaction), and magnetostatic energy contributions. This theory also predicts the formation of domain structures. The reader is referred to [17] for a recent survey on the topic. In the isothermal situation, the configuration of a rigid ferromagnetic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ is typically described by a magnetization $m : \Omega \to \mathbb{R}^d$ which vanishes if the temperature θ is above the so-called *Curie temperature* θ_c and no external magnetic field is applied.

On *microscopic level*, the magnetic *Gibbs energy* consists of five parts, namely an *anisotropy energy* $\int_\Omega \psi(m,\theta)\,\mathrm{d}x$, an *exchange energy* $\frac{1}{2}\int_\Omega \varepsilon|\nabla m(x)|^2\mathrm{d}x$ having a quantum-theoretical origin, the non-local *magnetostatic energy* $\frac{1}{2}\int_{\mathbb{R}^d}\mu_0|\nabla u_m(x)|^2\mathrm{d}x = \frac{1}{2}\int_\Omega m\cdot\nabla u_m\,\mathrm{d}x$, an *interaction energy* $-\int_\Omega h(x)\cdot m(x)\,\mathrm{d}x$ involving the outer magnetic field h and, finally, a *calorimetric term* $\int_\Omega \psi_0\,\mathrm{d}x$. In the anisotropic energy, we denoted ψ its density that will depend on the material properties and should exhibit crystallographic symmetry. Furthermore, ψ is supposed to be a nonnegative function, even in its first variable. In the magnetostatic energy, u_m is the magnetostatic potential related to m by $\mathrm{div}(\mu_0\nabla u_m - \chi_\Omega m) = 0$ arising from simplified magnetostatic Maxwell equations. Here $\chi_\Omega : \mathbb{R}^d \to \{0,1\}$ denotes the characteristic function of Ω and μ_0 is the permeability of vacuum. Moreover, it shall be noted that, due to the quantum mechanical origin, ε in the exchange energy is very small from the macroscopic point of view.

A widely accepted model for steady-state isothermal configurations is due to Landau and Lifshitz [20, 21] (see e.g. Brown [5] or Hubert and Schäfer [13]), relying on a minimum-of-Gibbs'-energy principle with θ as a parameter, i.e.

$$\left.\begin{aligned}
\text{minimize} \quad & G_\varepsilon(m) := \int_\Omega \left(\psi(m,\theta) + \frac{1}{2}m\cdot\nabla u_m + \frac{\varepsilon}{2}|\nabla m|^2 - h\cdot m\,\mathrm{d}x\right)\mathrm{d}x \\
\text{subject to} \quad & \mathrm{div}(\mu_0\nabla u_m - \chi_\Omega m) = 0 \ \text{ in } \mathbb{R}^d\,, \\
& m \in H^1(\Omega;\mathbb{R}^d), \ \ u_m \in H^1(\mathbb{R}^d),
\end{aligned}\right\} \tag{1.1}$$

where the anisotropy energy ψ is considered in the form

$$\psi(m, \theta) := \phi(m) + a_0(\theta - \theta_c)|m|^2 - \psi_0(\theta), \tag{1.2}$$

where a_0 determines the intensity of the thermo-magnetic-coupling. To see a paramagnetic state above Curie temperature θ_c, one should consider $a_0 > 0$. The *isothermal part of the anisotropy energy density* $\phi : \mathbb{R}^d \to [0, \infty)$ typically consists of two components $\phi(m) = \phi_{\mathrm{poles}}(m) + b_0|m|^4$, where $\phi_{\mathrm{poles}}(m)$ is chosen in such a way to attain its minimum value (typically zero) precisely on lines $\{ts_\alpha; \ t \in \mathbb{R}\}$, where each $s_\alpha \in \mathbb{R}^d$, $|s_\alpha| = 1$ determines an *axis of easy magnetization*. Typical examples are $\alpha = 1$ for uni-axial magnets and $1 \leq \alpha \leq 3$ or $1 \leq \alpha \leq 4$ for cubic magnets. On the other hand, $b_0|m|^4$ is used to assure that, for $\theta < \theta_c$, $\psi(\cdot, \theta)$ is minimized at ts_α for $|t|^2 = (\theta_c - \theta)a_0/(2b_0)$. Such energy has already been used in [31]. For $\varepsilon > 0$, the exchange energy $\varepsilon|\nabla u|^2$ guarantees that the problem (1.1) has a solution m_ε. Zero-temperature limits of this model consider, in addition, that the minimizers to (1.1) are constrained to be valued on the sphere with the radius $\sqrt{a_0\theta_c/(2b_0)}$ and were investigated, e.g., by Choksi and Kohn [7], DeSimone [9], James and Kinderlehrer [14], James and Müller [15], Pedregal [28, 29], Pedregal and Yan [30] and many others.

For ε small, minimizers m_ε of (1.1) typically exhibit fast spatial oscillations, a so-called *fine structure*. Indeed, the anisotropy energy, which forces magnetization vectors to be aligned with the easy axis (axes), competes with the magnetostatic energy preferring divergence-free magnetization fields. If the exchange energy term is neglected, and this is a justified simplification of the functional for large ferromagnets [9], nonexistence of a minimum for uniaxial ferromagnets can be expected and was shown in [14] for the zero external field h and zero temperature. Hence, various concepts of relaxation (in the sense of variational calculus) were introduced in order to cope with this phenomenon. The idea is to capture the limiting behavior of minimizing sequences of $G_\varepsilon(m)$ as $\varepsilon \to 0$. This leads to a relaxed problem (1.3) involving so-called Young measures ν's which describe the relevant "mesoscopical" character of the fine structure of m. We call this "limit" a *microstructure*.

It can be proved [9, 28] that this limit configuration (ν, u_m) solves the following minimization problem involving "mesoscopical" Gibbs' energy G:

$$\left.\begin{aligned}
\text{minimize} \quad & G(\nu, m) := \int_\Omega \left(\psi \bullet \nu + \frac{1}{2}m\cdot\nabla u_m - h\cdot m\right) \mathrm{d}x \\
\text{subject to} \quad & \mathrm{div}\big(\mu_0 \nabla u_m - \chi_\Omega m\big) = 0 \quad \text{on } \mathbb{R}^d, \\
& m = \mathrm{id} \bullet \nu \quad\quad\quad\quad\quad \text{on } \Omega, \\
& \nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d), \quad m \in L^p(\Omega; \mathbb{R}^d), \quad u_m \in H^1(\mathbb{R}^d) \ ,
\end{aligned}\right\} \tag{1.3}$$

where the "momentum" operator " $\bullet$ " is defined by $[f \bullet \nu](x) := \int_{\mathbb{R}^d} f(s)\nu_x(\mathrm{d}s)$, $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$ denotes the identity and $\nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d)$. Here, the set of *Young measures* $\mathscr{Y}^p(\Omega; \mathbb{R}^d) \subset L_{\mathrm{w}}^\infty(\Omega; \mathcal{M}(\mathbb{R}^d)) \cong L^1(\Omega; C_0(\mathbb{R}^d))^*$ is the set of all weakly measurable essentially bounded mappings $x \mapsto \nu_x : \Omega \to \mathcal{M}(\mathbb{R}^d) \cong C_0(\mathbb{R}^d)^*$ such that $\int_\Omega \int_{\mathbb{R}^d} |s|^p \nu_x(\mathrm{d}s)\mathrm{d}x < +\infty$; here $C_0(\mathbb{R}^d)$ denotes the set of continuous functions with compact support and thus $\mathcal{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$, and the adjective "weakly measurable" means that $v \bullet \nu$ is Lebesgue measurable for any $v \in C_0(\mathbb{R}^d)$. A natural embedding $i : L^p(\Omega; \mathbb{R}^d) \to \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ of a magnetization m is a Young measure $\nu = i(m)$ defined by $\nu_x = \delta_{m(x)}$ with δ_s denoting the Dirac measure at $s \in \mathbb{R}^d$. We say that a sequence $\{\nu^k\}_{k\in\mathbb{N}} \subset \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ converges weakly* to ν if

$$\forall f \in L^1(\Omega; C_0(\mathbb{R}^d)) : \quad \lim_{k\to\infty} \int_\Omega f \bullet \nu^k \mathrm{d}x \ = \ \int_\Omega f \bullet \nu \, \mathrm{d}x \tag{1.4}$$

The set $\mathscr{Y}^p(\Omega; \mathbb{R}^d)$ is convex, metrizable, compact, and contains weak*-densely the set of magnetizations $m \in L^p(\Omega; \mathbb{R}^d)$ if embedded via i. One can thus call $\mathscr{Y}^p(\Omega; \mathbb{R}^d)$ a *convex compactification* of the set of admissible magnetizations, cf. also [33, Chapter 3]. The so-called *relaxed problem* (1.3) can then be understood simply as a continuous extension of the original problem

(1.1) considered for $\varepsilon = 0$. Let us note that the problem (1.3) has a convex structure. Moreover, it captures the *multiscale* character of the problem. The Young measure solving (1.3) encodes limiting oscillating behavior of minimizing sequences of (1.1) while its first moment, the magnetization, resolves the macroscopic magnetization m. Due to these properties we call the model in (1.3) *mesoscopic*. An equivalent way how to relax (1.1) is to replace ψ by its convex envelope. The drawback for numerical calculations is that one needs to know the convex envelope explicitly. There were many attempts to design numerical schemes for both (1.1), as well as for (1.3) in the zero-temperature situation; cf. e.g. [6, 16, 22]. Departing from (1.3) and following the ideas of [26], a model of the isothermal rate-independent evolution exhibiting hysteretic response was proposed and analyzed in [35, 36].

Our motivation is to merge concepts of relaxation that can successfully fight with multiscale character of the problem with recent ideas to build thermodynamically consistent mesoscopic models in anisothermal situations. A closely related thermodynamically consistent model on the microscopic level was introduced in [31] to ferro/para magnetic transition. Another related microscopic model with a prescribed temperature field was investigated in [1]. The goal is to develop a model that would be supported by rigorous analysis and would allow for computationally efficient numerical implementation like in [16, 18, 19] where such a model was used in the isothermal variant.

This paper is structured as follows: In Section 2 we introduce the concept of general standard materials and handle thermodynamics, in Section 3 we give a weak formulation of the model equations, in Section 4 we give a time-discretization of the model equations and prove a priori estimates, in Section 5 we then prove convergence of the discrete solutions. Finally in Section 6, we present a generalization of the model, where we allow for a weaker dissipation so that at least some internal parameters can be considered evolving in a purely rate-independent manner.

2. Evolution problem, dissipation, mesoscopic Gibbs free energy

If the external magnetic field h varies during a time interval $[0, T]$, $T > 0$, the energy of the system as well as the magnetization evolve, too. Change of magnetization may cause energy dissipation. As the magnetization is the first moment of the Young measure we relate the dissipation on the mesoscopic level to temporal changes of some moments of ν and consider these moments as separate variables. This approach was already used in micromagnetics in [35, 36] and proved to be useful also in modeling of dissipation in shape memory materials, see e.g. [24]. In view of (1.2), we restrict ourselves to the first two moments defining $\lambda = (\lambda_1, \lambda_2)$ giving rise to the constraint

$$\lambda = L \bullet \nu \,, \qquad \text{where } L(m) := (m, |m|^2) \tag{2.1}$$

and consider the specific dissipation potential

$$\zeta(\dot{\lambda}) := \delta_S^*(\dot{\lambda}) + \frac{\epsilon}{q} |\dot{\lambda}|^q, \qquad q \geq 2, \tag{2.2}$$

where $\delta_S^* : \mathbb{R}^{d+1} \to \mathbb{R}^+$ is the Legendre-Fenchel conjugate function to the indicator function $\delta_S : \mathbb{R}^{d+1} \to \{0, +\infty\}$ to a convex bounded neighborhood of 0. The set S determines activation threshold for the evolution of λ. The function $\delta_S^* \geq 0$ is convex and degree-1 positively homogeneous with $\delta_S^*(0) = 0$. In fact, the first term describes purely hysteretic losses, which are rate-independent (the contribution of which we consider dominant) and the second term models rate-dependent dissipation.

In view of (1.2)–(1.3), the specific mesoscopic Gibbs free energy, expressed in terms of ν, λ and θ, reads as

$$g(t,\nu,\lambda,\theta) := \phi \bullet \nu + (\theta-\theta_{\mathrm{c}})\vec{a}\cdot\lambda - \psi_0(\theta) + \frac{1}{2}m\cdot h_{\mathrm{dem}} - h(t)\cdot m \qquad (2.3\mathrm{a})$$

$$\text{with} \quad m = \mathrm{id} \bullet \nu \quad \text{and} \quad h_{\mathrm{dem}} = \nabla u_m, \qquad (2.3\mathrm{b})$$

where we denoted $\vec{a} := (0,\ldots,0,a_0)$ with a_0 from (1.2) and, of course, u_m again from (1.1),which makes g non-local.

In what follows, we relax the constraint (2.1) by augmenting the total Gibbs free energy (i.e. g integrated over Ω) by the term $\frac{\varkappa}{2}\|\lambda - L \bullet \nu\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ with (presumably large) $\varkappa \in \mathbb{R}^+$ and with $H^{-1}(\Omega;\mathbb{R}^{d+1}) \cong H^1_0(\Omega;\mathbb{R}^{d+1})^*$. Thus, λ's no longer exactly represent the "macroscopical" momenta of the magnetization but rather are in a position of a *phase field*. Let us choose a specific norm on $H^{-1}(\Omega;\mathbb{R}^{d+1})$ as $\|f\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})} := \|\chi\|_{L^2(\Omega;\mathbb{R}^{d+1})}$ with $\chi \in L^2_{\mathrm{grad}}(\Omega;\mathbb{R}^d)$ defined uniquely by $\mathrm{div}\,\chi = \lambda - L \bullet \nu$ where $L^2_{\mathrm{grad}}(\Omega;\mathbb{R}^d) := \{\nabla v;\ v \in H^1_0(\Omega;\mathbb{R}^{d+1})\}$. In other words, $\|f\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})} = \|\nabla\Delta^{-1}f\|_{L^2(\Omega;\mathbb{R}^{d+1})}$ with Δ^{-1} meaning the inverse of the Dirichlet boundary-value problem for the Laplacean $\Delta : H^1_0(\Omega;\mathbb{R}^{d+1}) \to H^{-1}(\Omega;\mathbb{R}^{d+1})$. We define the mesoscopic Gibbs free energy $\mathscr{G}$ by

$$\mathscr{G}(t,\nu,\lambda,\theta) := \int_\Omega \left(g(t,\nu,\lambda,\theta) + \frac{\varkappa}{2}|\nabla\Delta^{-1}(\lambda - L \bullet \nu)|^2\right)\mathrm{d}x \qquad (2.4)$$

Notice that the H^{-1}norm defined above is equivalent to the standard H^{-1}-norm defined as $\|f\|_{-1} := \sup_{v \in W^{1,2}_0(\Omega;\mathbb{R}^{d+1}),\ v\neq 0} \langle f,v\rangle / \|v\|_{W^{1,2}_0(\Omega;\mathbb{R}^{d+1})}$. Indeed, if $h \in W^{1,2}_0(\Omega;\mathbb{R}^{d+1})$ solves $-\Delta h = f$ and $\|v\|_{W^{1,2}_0(\Omega;\mathbb{R}^{d+1})} := \|\nabla v\|_{L^2(\Omega;\mathbb{R}^{d+1})}$ then

$$\|f\|_{-1} = \sup_{v \in W^{1,2}_0(\Omega;\mathbb{R}^{d+1}),\ v\neq 0} \|v\|^{-1}_{W^{1,2}_0(\Omega;\mathbb{R}^{d+1})}\int_\Omega \nabla h \cdot \nabla v\,\mathrm{d}x \geq \beta\|\nabla h\|_{L^2(\Omega;\mathbb{R}^{d+1})}\ ,$$

where $\beta > 0$ is the ellipticity constant of $B[h,v] := \int_\Omega \nabla h \cdot \nabla v\,\mathrm{d}x$. On the other hand,

$$\|f\|_{-1} = \sup_{v \in W^{1,2}_0(\Omega;\mathbb{R}^{d+1}),\ v\neq 0} \|v\|^{-1}_{W^{1,2}_0(\Omega;\mathbb{R}^{d+1})}\int_\Omega \nabla h \cdot \nabla v\,\mathrm{d}x \leq \|\nabla h\|_{L^2(\Omega;\mathbb{R}^{d+1})}$$

by the Cauchy-Schwartz inequality.

We shall give a certain justification of the penalized model at the end of the section. Yet, we should emphasize that we will consider $\varkappa$ fixed thorough this article.

The value of the internal parameter may influence the magnetization of the system (and vice versa) and, on the other hand, dissipated energy may influence the temperature of the system, which, in turn, may affect the internal parameter. In order to capture these effects we employ the concept of general standard materials [12] known from continuum mechanics and couple our micromagnetic model with the entropy balance with the rate of dissipation on the right-hand side; cf. (2.6). Then ν is considered evolving purely quasistatically according to the minimization principle of the Gibbs energy $\mathcal{G}(t,\cdot,\lambda,\theta)$ while the "dissipative" variable λ is considered as governed by the *flow rule* in the form:

$$\partial\zeta(\dot{\lambda}) = \partial_\lambda g(t,\nu,\lambda,\theta) \qquad (2.5)$$

with $\partial\zeta$ denoting the subdifferential of the convex functional $\zeta(\cdot)$ and similarly $\partial_\lambda g$ is the subdifferential of the convex functional $g(t,\nu,\cdot,\theta)$. In our specific choice, (2.5) takes the form $\partial\delta^*_S(\dot{\lambda}) + \epsilon|\dot{\lambda}|^{q-2}\dot{\lambda} + (\theta-\theta_{\mathrm{c}})\vec{a} \ni \varkappa\Delta^{-1}(\lambda - L \bullet \nu)$. Furthermore, we define the specific *entropy s* by the standard Gibbs relation for entropy, i.e. $s = -g'_\theta(t,\nu,\lambda,\theta)$, and write the *entropy equation*

$$\theta\dot{s} + \mathrm{div}\,j = \xi(\dot{\lambda}) = \quad \text{heat production rate,} \qquad (2.6)$$

where j is the heat flux. In view of (2.2),

$$\xi(\dot{\lambda}) = \partial\zeta(\dot{\lambda})\cdot\dot{\lambda} = \delta_S^*(\dot{\lambda}) + \epsilon|\dot{\lambda}|^q. \tag{2.7}$$

We assume j is the heat flux assumed governed by the *Fourier law*

$$j = -\mathbb{K}\nabla\theta \tag{2.8}$$

with a *heat-conductivity* tensor $\mathbb{K} = \mathbb{K}(\lambda,\theta)$. Now, since $s = -g_\theta'(t,\nu,\lambda,\theta) = -g_\theta'(\lambda,\theta)$, it holds $\theta\dot{s} = -\theta g_\theta''(\lambda,\theta)\dot{\theta} - \theta g_{\theta\lambda}''\dot{\lambda}$. Using also $g_{\theta\lambda}'' = \vec{a}$, we may reformulate the entropy equation (2.6) as *the heat equation*

$$c_v(\theta)\dot{\theta} - \mathrm{div}(\mathbb{K}(\lambda,\theta)\nabla\theta) = \delta_S^*(\dot{\lambda}) + \epsilon|\dot{\lambda}|^q + \vec{a}\cdot\theta\dot{\lambda} \quad \text{with} \ \ c_v(\theta) = -\theta g_\theta''(\theta), \tag{2.9}$$

where c_v is the specific *heat capacity*.

Altogether, we can formulate our problem as

$$
\begin{aligned}
\text{minimize} \quad & \int_\Omega\Big(\phi\bullet\nu+(\theta(t))-\theta_c)\vec{a}\cdot\lambda(t)-\psi_0(\theta(t)) + \tfrac{1}{2}m\cdot h_{\mathrm{dem}} \\
& \qquad -h(t)\cdot m + \tfrac{\varkappa}{2}\big|\nabla\Delta^{-1}(\lambda(t)-L\bullet\nu)\big|^2\Big)\,\mathrm{d}x \\
\text{subject to} \quad & m = \mathrm{id}\bullet\nu, \quad h_{\mathrm{dem}} = \nabla u_m \qquad \text{on}\ \Omega, \\
& \mathrm{div}\big(\mu_0\nabla u_m - \chi_\Omega m\big) = 0 \qquad \text{on}\ \mathbb{R}^d, \\
& \nu\in\mathscr{Y}^p(\Omega;\mathbb{R}^d),\ m\in L^p(\Omega;\mathbb{R}^d),\ u_m\in H^1(\mathbb{R}^d),
\end{aligned}
\left.\vphantom{\begin{aligned}1\\1\\1\\1\\1\end{aligned}}\right\} \ \text{for}\ t\in[0,T], \tag{2.10a}
$$

$$\partial\delta_S^*(\dot{\lambda}) + \epsilon|\dot{\lambda}|^{q-2}\dot{\lambda} + (\theta-\theta_c)\vec{a} \ni \varkappa\Delta^{-1}(\mathrm{div}\,\lambda - L\bullet\nu) \qquad \text{in}\ Q := [0,T]\times\Omega, \tag{2.10b}$$

$$c_v(\theta)\dot{\theta} - \mathrm{div}(\mathbb{K}(\lambda,\theta)\nabla\theta) = \delta_S^*(\dot{\lambda}) + \epsilon|\dot{\lambda}|^q + \vec{a}\cdot\dot{\lambda}\theta \qquad \text{in}\ Q, \tag{2.10c}$$

$$\big(\mathbb{K}(\lambda,\theta)\nabla\theta\big)\cdot n + b\theta = b\theta_{\mathrm{ext}} \qquad \text{on}\ \Sigma := [0,T]\times\Gamma, \tag{2.10d}$$

where we accompanied the heat equation (2.10c) by the Robin-type boundary conditions with n denoting the outward unit normal to the boundary Γ, and with $b \in L^\infty(\Gamma)$ a phenomenological heat-transfer coefficient and θ_{ext} an external temperature, both assumed non-negative.

Next we shall transform (2.10c) by a so-called *enthalpy transformation*, which simplifies the analysis below. For this, let us introduce a new variable w, called *enthalpy*, by

$$w = \widehat{c}_v(\theta) = \int_0^\theta c_v(r)\mathrm{d}r, \tag{2.11}$$

It is natural to assume c_v positive, hence $\widehat{c}_v$ is, for $w \geq 0$ increasing and thus invertible. Therefore, denote

$$\Theta(w) := \begin{cases} \widehat{c}_v^{-1}(w) & \text{if } w \geq 0 \\ 0 & \text{if } w < 0 \end{cases}$$

and note that, in the physically relevant case when $\theta \geq 0$, $\theta = \Theta(w)$. Thus writing the heat flux in terms of w gives

$$\mathbb{K}(\lambda,\theta)\nabla\theta = \mathbb{K}\big(\lambda,\Theta(w)\big)\nabla\Theta(w) = \mathcal{K}(\lambda,w)\nabla w \quad \text{where} \ \ \mathcal{K}(\lambda,w) := \frac{\mathbb{K}(\lambda,\Theta(w))}{c_v(\Theta(w))}. \tag{2.12}$$

Moreover, the terms $(\Theta(w(t))-\theta_c)\vec{a}\cdot\lambda(t)$ and $\psi_0(\theta(t))$ obviously do not play any role in the minimization (2.10a) and can be omitted. Thus we may rewrite (2.10a) in terms of w as follows:

$$\left.\begin{aligned}
\text{minimize} \quad & \int_\Omega \Big(\phi \bullet \nu + \frac{1}{2} m \cdot h_{\text{dem}} - h(t) \cdot m + \frac{\varkappa}{2} \big|\nabla \Delta^{-1}(\lambda(t) - L \bullet \nu)\big|^2\Big)\, \mathrm{d}x \\
\text{subject to} \quad & m = \mathrm{id} \bullet \nu, \quad h_{\text{dem}} = \nabla u_m \qquad \text{on } \Omega, \\
& \mathrm{div}\big(\mu_0 \nabla u_m - \chi_\Omega m\big) = 0 \qquad \text{on } \mathbb{R}^d, \\
& \nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d), \ m \in L^p(\Omega; \mathbb{R}^d), \ u_m \in H^1(\mathbb{R}^d),
\end{aligned}\right\} \text{ for } t \in [0,T], \text{(2.13a)}$$

$$\partial \delta_S^*(\dot\lambda) + \epsilon |\dot\lambda|^{q-2}\dot\lambda + \big(\Theta(w) - \theta_c\big)\vec{a} \ni \varkappa \Delta^{-1}(\lambda - L \bullet \nu) \qquad\qquad \text{in } Q, \qquad\qquad \text{(2.13b)}$$

$$\dot w - \mathrm{div}(\mathcal{K}(\lambda, w)\nabla w) = \delta_S^*(\dot\lambda) + \epsilon|\dot\lambda|^q + \vec{a}\cdot\Theta(w)\dot\lambda \qquad\qquad \text{in } Q, \qquad\qquad \text{(2.13c)}$$

$$\big(\mathcal{K}(\lambda, w)\nabla w\big)\cdot n + b\Theta(w) = b\theta_{\text{ext}} \qquad\qquad\qquad \text{on } \Sigma. \qquad\qquad \text{(2.13d)}$$

Eventually, we complete this transformed system by the initial conditions

$$\nu(0,\cdot) = \nu_0, \qquad \lambda(0,\cdot) = \lambda_0, \qquad w(0,\cdot) = w_0 := \widehat{c}_v(\theta_0) \qquad \text{on } \Omega, \qquad (2.14)$$

where (ν_0, λ_0) is the initial microstructure assumed to solve (2.13a) and the phase field, and θ_0 is the initial temperature. Note also that, by prescribing ν_0, we also prescribe the initial magnetization $m_0 = \mathrm{id} \bullet \nu_0$ and magnetic potential u_{m_0}.

2.1. Justification of the penalization concept

Recall that in the model (2.10) we gave up the constraint $\lambda = L \bullet \nu$ and only included a penalization term $\frac{\varkappa}{2}\|\lambda - L \bullet \nu\|_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$ in the Gibbs free energy. To justify this approach, we show that in some particular situations, namely in the static case and also in the iso-thermal rate-independent (with a small modification) case, solutions of the penalized model converge to solutions of the original model that satisfy $\lambda = L \bullet \nu$ as $\varkappa \to \infty$. We shall also give some heuristic ideas, why a similar limit passage should be possible even in (2.10), however a rigorous proof is beyond the scope of this paper.

The static case:

Let us consider an analogical problem to (1.3) that includes also a penalization term and where we also use the form of the Gibbs free energy as in (2.3a) with θ given, i.e.

$$\left.\begin{aligned}
\text{minimize} \quad & \int_\Omega \Big(\phi \bullet \nu + (\theta - \theta_c)\vec{a}\cdot\lambda - \psi_0(\theta) + \frac{1}{2} m \cdot \nabla u_m - h(t)\cdot m + \frac{\varkappa}{2}|\nabla\Delta^{-1}(\lambda - L \bullet \nu)|^2\Big)\, \mathrm{d}x \\
\text{subject to} \quad & \mathrm{div}\big(\mu_0 \nabla u_m - \chi_\Omega m\big) = 0 \quad \text{on } \mathbb{R}^d, \\
& m = \mathrm{id} \bullet \nu \qquad\qquad\qquad \text{on } \Omega, \\
& \nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d), \quad \lambda \in H^{-1}(\Omega; \mathbb{R}^{d+1}), \quad m \in L^p(\Omega; \mathbb{R}^d), \quad u_m \in H^1(\mathbb{R}^d).
\end{aligned}\right\} \text{(2.15)}$$

Let us denote $(\lambda_\varkappa, \nu_\varkappa) \in H^{-1}(\Omega; \mathbb{R}^{d+1}) \times \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ the solutions to (2.15). Let us then show that they (in terms of a subsequence) converge weakly* in $H^{-1}(\Omega; \mathbb{R}^{d+1}) \times \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ to the solutions of

$$\left.\begin{aligned}
\text{minimize} \quad & \int_\Omega \Big(\phi \bullet \nu + (\theta - \theta_c)\vec{a}\cdot\lambda - \psi_0(\theta) + \frac{1}{2} m \cdot \nabla u_m - h(t)\cdot m\Big)\, \mathrm{d}x \\
\text{subject to} \quad & \mathrm{div}\big(\mu_0 \nabla u_m - \chi_\Omega m\big) = 0 \quad \text{on } \mathbb{R}^d, \\
& m = \mathrm{id} \bullet \nu, \ \lambda = L \bullet \nu \qquad \text{on } \Omega, \\
& \nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d), \quad \lambda \in H^{-1}(\Omega; \mathbb{R}^{d+1}), \quad m \in L^p(\Omega; \mathbb{R}^d), \quad u_m \in H^1(\mathbb{R}^d)
\end{aligned}\right\} \text{(2.16)}$$

Namely, it easy to see, from coercivity of ϕ and by simply testing (2.15) by any $(\hat\lambda, \hat\nu)$ such that $\hat\lambda = L \bullet \hat\nu$, that $\int_\Omega |\cdot|^p \bullet \nu_\varkappa \mathrm{d}x$ is bounded uniformly with respect to $\varkappa$; here p corresponds to the growth of ϕ, cf. (3.5a). Hence also $\|L \bullet \nu_\varkappa\|_{L^2(\Omega; \mathbb{R}^{d+1})}$ and in turn also $\|L \bullet \nu_\varkappa\|_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$ are uniformly bounded with respect to $\varkappa$.

Also, by the same test as above, we get that $\varkappa\|\lambda_\varkappa - L\bullet\nu_\varkappa\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ and thus also $\|\lambda_\varkappa\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ are bounded independently of $\varkappa$.

Hence, exploiting standard selection principles, we find a pair $(\nu, \lambda) \in \mathscr{Y}^p(\Omega;\mathbb{R}^d) \times H^{-1}(\Omega;\mathbb{R}^{d+1})$ such that (in terms of a not-relabeled subsequence) $\nu_\varkappa \overset{*}{\rightharpoonup} \nu$ in $\mathscr{Y}^p(\Omega;\mathbb{R}^d)$ and $\lambda_\varkappa \rightharpoonup \lambda$ in $H^{-1}(\Omega;\mathbb{R}^{d+1})$. Also, as $\varkappa\|\lambda_\varkappa - L\bullet\nu_\varkappa\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ is bounded independently of $\varkappa$, necessarily $\lambda = L\bullet\nu$ holds for the weak limits.

Then thanks to the weak-lower semi-continuity we have that

$$
\int_\Omega \left(\phi\bullet\nu + (\theta-\theta_c)\vec{a}\cdot\lambda - \psi_0(\theta) + \frac{1}{2}m\cdot\nabla u_m - h(t)\cdot m \right) \mathrm{d}x
$$

$$
\leq \liminf_{\varkappa\to\infty} \int_\Omega \left(\phi\bullet\nu_\varkappa + (\theta-\theta_c)\vec{a}\cdot\lambda_\varkappa - \psi_0(\theta) + \frac{1}{2}m_\varkappa\cdot\nabla u_{m_\varkappa} - h(t)\cdot m_\varkappa \right) \mathrm{d}x
$$

$$
\leq \liminf_{\varkappa\to\infty} \int_\Omega \left(\phi\bullet\nu_\varkappa + (\theta-\theta_c)\vec{a}\cdot\lambda_\varkappa - \psi_0(\theta) + \frac{1}{2}m_\varkappa\cdot\nabla u_{m_\varkappa} - h(t)\cdot m_\varkappa \right) \mathrm{d}x + \frac{\varkappa}{2}\|\lambda_\varkappa - L\bullet\nu_\varkappa\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}
$$

$$
\leq \liminf_{\varkappa\to\infty} \int_\Omega \left(\phi\bullet\hat{\nu} + (\theta-\theta_c)\vec{a}\cdot\hat{\lambda} - \psi_0(\theta) + \frac{1}{2}\hat{m}\nabla u_{\hat{m}} - h(t)\cdot\hat{m} \right) \mathrm{d}x + \frac{\varkappa}{2}\|\hat{\lambda} - L\bullet\hat{\nu}\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}
$$

$$
= \int_\Omega \left(\phi\bullet\hat{\nu} + (\theta-\theta_c)\vec{a}\cdot\hat{\lambda} - \psi_0(\theta) + \frac{1}{2}\hat{m}\nabla u_{\hat{m}} - h(t)\cdot\hat{m} \right) \mathrm{d}x
$$

for any $(\hat{\lambda}, \hat{\nu})$ such that $\hat{\lambda} = L\bullet\hat{\nu}$, which shows that (λ, ν) is a solution to (2.16).

The rate-independent isothermal case:

When considering the rate-independent case we formally set $\epsilon = 0$ in (2.2), i.e. assume that the dissipation potential is equal to δ^*_S. Also, since now the dissipation potential yields less regularity on λ we have to alter the specific mesoscopic Gibbs free energy and to add a regularization term $\gamma|\nabla\lambda|^2$; let us therefore denote

$$
g_{\mathrm{RI}}(t, \nu, \lambda) = g(t, \nu, \lambda, \theta) + \gamma|\nabla\lambda|^2,
$$

again for some θ fixed. Now, as the temperature is not a variable in this context, the system of governing equations (2.10) reduces to (with $g_{\mathrm{RI}}(t, \nu, \lambda)$ replacing $g(t, \nu, \lambda, \theta)$)

$$
\left.
\begin{aligned}
\text{minimize} \quad & \int_\Omega \left(g_{\mathrm{RI}}(t, \nu, \lambda,) + \frac{\varkappa}{2}\left|\nabla\Delta^{-1}(\lambda(t) - L\bullet\nu)\right|^2 \right) \mathrm{d}x \\
\text{subject to} \quad & m = \mathrm{id}\bullet\nu, && \text{on } \Omega, \\
& \mathrm{div}\big(\mu_0\nabla u_m - \chi_\Omega m\big) = 0 && \text{on } \mathbb{R}^d, \\
& \nu \in \mathscr{Y}^p(\Omega;\mathbb{R}^d), \ m \in L^p(\Omega;\mathbb{R}^d), \ u_m \in H^1(\mathbb{R}^d),
\end{aligned}
\right\} \text{ for } t\in[0,T],
\qquad (2.17a)
$$

$$
\partial\delta^*_S(\dot{\lambda}) + (\theta-\theta_c)\vec{a} + 2\gamma\mathrm{div}\nabla\lambda + \varkappa\Delta^{-1}(\mathrm{div}\,(\lambda - L\bullet\nu)) \ni 0 \qquad \text{in } Q := [0,T]\times\Omega, \qquad (2.17b)
$$

and we recover, apart from the penalization, a similar model as in [18, 19]. Since we are considering the rate-independent case, a suitable weak formulation of (2.17), which in fact - due to the convexity of the problem - is equivalent to the standard weak formulation, is the so-called energetic formulation, cf. e.g. [26]. Then we shall call $(\lambda_\varkappa, \nu_\varkappa) \in L^\infty([0,T]; W^{1,2}(\Omega;\mathbb{R}^{d+1})) \cap \mathrm{BV}([0,T]; L^1(\Omega;\mathbb{R}^{d+1})) \times (\mathscr{Y}^p(\Omega;\mathbb{R}^d))^{[0,T]}$ an energetic solution of (2.17) if they satisfy (we

included initial conditions here already)

$$\int_\Omega \Big(g_{\mathrm{RI}}(t, \nu_\varkappa(t), \lambda_\varkappa(t)) + \frac{\varkappa}{2}\big|\nabla\Delta^{-1}(\lambda_\varkappa(t) - L\bullet\nu_\varkappa(t))\big|^2\Big)\mathrm{d}x$$

$$\leq \int_\Omega \Big(g_{\mathrm{RI}}(t, \hat\nu, \hat\lambda) + \frac{\varkappa}{2}\big|\nabla\Delta^{-1}(\hat\lambda - L\bullet\hat\nu)\big|^2 + \delta_S^*(\hat\lambda - \lambda_\varkappa)\Big)\mathrm{d}x$$

$$\text{for all } (\hat\lambda, \hat\nu) \in W^{1,2}(\Omega;\mathbb{R}^{d+1}) \times \mathscr{Y}^p(\Omega;\mathbb{R}^d) \text{ and all } t \in [0,T] \,, \qquad (2.18a)$$

$$\int_\Omega \Big(g_{\mathrm{RI}}(T, \nu_\varkappa(T), \lambda_\varkappa(T)) + \frac{\varkappa}{2}\big|\nabla\Delta^{-1}(\lambda_\varkappa(T) - L\bullet\nu_\varkappa(T))\big|^2\Big)\mathrm{d}x + \mathrm{Var}_{\delta_S^*}(\lambda;0,T)$$

$$\leq \int_\Omega g_{\mathrm{RI}}(0, \nu_0, \lambda_0)\mathrm{d}x + \int_0^T \int_\Omega [g_{\mathrm{RI}}]_t'(s, \nu_\varkappa(s))\mathrm{d}x\mathrm{d}s, \qquad (2.18b)$$

$$\lambda_\varkappa(0) = \lambda_0 \in W^{1,2}(\Omega;\mathbb{R}^{d+1}), \quad \nu_\varkappa(0) = \nu_0 \in \mathscr{Y}^p(\Omega;\mathbb{R}^d), \quad \lambda_0 = L\bullet\nu_0, \qquad (2.18c)$$

with $\mathrm{Var}_f(x;0,T)$ the space integral of the variation of f between 0 and T. Let us now show that energetic solutions $(\lambda_\varkappa, \nu_\varkappa)$ of (2.17) converge in $L^\infty([0,T]; W^{1,2}(\Omega;\mathbb{R}^{d+1})) \cap \mathrm{BV}([0,T]; L^1(\Omega;\mathbb{R}^{d+1})) \times (\mathscr{Y}^p(\Omega;\mathbb{R}^d))^{[0,T]}$ (at least in terms of a subsequence) for $\varkappa \to \infty$ to (λ, ν) satisfying

$$\int_\Omega g_{\mathrm{RI}}(t, \nu(t), \lambda(t))\mathrm{d}x \leq \int_\Omega \big(g_{\mathrm{RI}}(t, \hat\nu, \hat\lambda) + \delta_S^*(\hat\lambda - \lambda)\big)\mathrm{d}x$$

$$\text{for all } (\hat\lambda, \hat\nu) \in W^{1,2}(\Omega;\mathbb{R}^{d+1}) \times \mathscr{Y}^p(\Omega;\mathbb{R}^d) \text{ such that } \hat\lambda = L\bullet\hat\nu \text{ and all } t \in [0,T] \,, \qquad (2.19a)$$

$$\int_\Omega g_{\mathrm{RI}}(T, \nu(T), \lambda(T))\mathrm{d}x + \mathrm{Var}_{\delta_S^*}(\lambda;0,T) \leq \int_\Omega g_{\mathrm{RI}}(0, \nu_0, \lambda_0)\mathrm{d}x + \int_Q [g_{\mathrm{RI}}]_t'(s, \nu(s))\mathrm{d}x\mathrm{d}s,$$

$$(2.19b)$$

$$\lambda(0) = \lambda_0 \in W^{1,2}(\Omega;\mathbb{R}^{d+1}), \quad \nu(0) = \nu_0 \in \mathscr{Y}^p(\Omega;\mathbb{R}^d), \quad \lambda_0 = L\bullet\nu_0, \qquad (2.19c)$$

i.e. the energetic formulation of

$$\left.\begin{array}{ll} \text{minimize} & \displaystyle\int_\Omega g_{\mathrm{RI}}(t, \nu, \lambda)\,\mathrm{d}x \\[2mm] \text{subject to} & m = \mathrm{id}\bullet\nu, \quad \lambda = L\bullet\nu, \qquad\qquad \text{on } \Omega, \\[1mm] & \mathrm{div}\big(\mu_0\nabla u_m - \chi_\Omega m\big) = 0 \qquad\qquad \text{on } \mathbb{R}^d, \\[1mm] & \nu \in \mathscr{Y}^p(\Omega;\mathbb{R}^d), \ m \in L^p(\Omega;\mathbb{R}^d), \ u_m \in H^1(\mathbb{R}^d), \end{array}\right\} \text{ for } t\in[0,T], \qquad (2.20a)$$

$$\partial\delta_S^*(\dot\lambda) + (\theta-\theta_{\mathrm{c}})\vec{a} + 2\gamma \div \nabla\lambda + \partial I_{\lambda = L\bullet\nu} \ni 0 \qquad \text{in } Q := [0,T]\times\Omega, \qquad (2.20b)$$

where $I_{\lambda = L\bullet\nu} = 0$, if $\lambda = L\bullet\nu$, and $+\infty$ otherwise.

The conjecture follows from the abstract paper [25]; here we give a very short sketch.

By similar tests as in the static case, it can be seen that $\|\lambda_\varkappa\|_{L^\infty([0,T]; W^{1,2}(\Omega;\mathbb{R}^{d+1}))\cap\mathrm{BV}([0,T]; L^1(\Omega;\mathbb{R}^{d+1}))}$, $\int_\Omega |\cdot|^p\bullet\nu_\varkappa\mathrm{d}x$ and $\varkappa\|\lambda_\varkappa - L\bullet\nu_\varkappa\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ are bounded independently of $\varkappa$.

Hence, by a slight modification of Helly's theorem [23, 25], there exists $\lambda \in L^\infty([0,T]; W^{1,2}(\Omega;\mathbb{R}^{d+1})) \cap \mathrm{BV}([0,T]; L^1(\Omega;\mathbb{R}^{d+1})$ and a not-relabeled subsequence of $\varkappa$ such that $\lambda_\varkappa(t) \to \lambda(t)$ for all $t \in [0,T]$ weakly in $W^{1,2}(\Omega;\mathbb{R}^{d+1})$ and hence strongly in $L^1(\Omega;\mathbb{R}^{d+1}))$.

Let us now fix $t \in [0,T]$. Then there exists a subsequence of $\varkappa$ (dependent on t) denoted $\varkappa_t$ and a $\nu \in \mathscr{Y}^p(\Omega;\mathbb{R}^d))$ such that $\nu_{\varkappa_t} \to \nu$ weakly* in $\mathscr{Y}^p(\Omega;\mathbb{R}^d)$. Similarly as in the static

case, using (2.18a) and weak lower semi-continuity, we get that

$$\int_\Omega g_{\mathrm{RI}}(t,\nu(t),\lambda(t))\mathrm{d}x \le \liminf_{\varkappa_t\to\infty}\int_\Omega g_{\mathrm{RI}}(t,\nu_{\varkappa_t}(t),\lambda_{\varkappa_t}(t))\mathrm{d}x + \frac{\varkappa_t}{2}\|\lambda_{\varkappa_t}-L\bullet\nu_{\varkappa_t}\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$$

$$\le \liminf_{\varkappa_t\to\infty}\int_\Omega\Big(g_{\mathrm{RI}}(t,\hat\nu,\hat\lambda) + \delta^*_S(\hat\lambda - \lambda_{\varkappa(t)}(t))\Big)\mathrm{d}x = \int_\Omega\Big(g_{\mathrm{RI}}(t,\hat\nu,\hat\lambda) + \delta^*_S(\hat\lambda - \lambda(t))\Big)\mathrm{d}x,$$

for all $(\hat\lambda,\hat\nu) \in W^{1,2}(\Omega;\mathbb{R}^{d+1}) \times \mathscr{Y}^p(\Omega;\mathbb{R}^d)$ such that $\hat\lambda = L\bullet\hat\nu$, i.e. we showed that (λ,ν) fulfills (2.19a). In the last line we exploited that δ^*_S is one-homogeneous and the *strong convergence* $\lambda_\varkappa(t) \to \lambda(t)$ in $L^1(\Omega;\mathbb{R}^{d+1}))$.

To see that (λ,ν) also fulfills (2.19b) we pass to the limit in (2.18b) exploiting only weak lower semi-continuity.

The thermally coupled case as exposed in (2.13):

As already mentioned, we only give a short heuristic sketch why the penalty approach is also justified in the case presented here, in particular we concentrate only on the limit passage in the minimization principle (2.13a) since this seems to be the most involved one. Let us for simplicity assume that $q = 2$.

Assume that $(\lambda_\varkappa,\nu_\varkappa) \in W^{1,2}([0,T];L^2(\Omega;\mathbb{R}^{d+1})) \times (\mathscr{Y}^p(\Omega;\mathbb{R}^d))^{[0,T]}$ (together with some $w_\varkappa \in L^1([0,T];W^{1,1}(\Omega))$, which is however irrelevant here) are weak solutions of (2.13) with an initial condition satisfying $\lambda_0 = L\bullet\nu_0$. Then (2.13a) yields, just by the chain rule, that

$$\int_\Omega\Big(\phi\bullet\nu_\varkappa(t)+\frac{1}{2}m_\varkappa(t)\cdot\nabla u_{m_\varkappa}(t) - h(t)\cdot m_\varkappa(t) + \frac{\varkappa}{2}\big|\nabla\Delta^{-1}(\lambda_\varkappa(t)-L\bullet\nu_\varkappa)\big|^2\Big)\mathrm{d}x$$

$$= \int_\Omega\Big(\phi\bullet\nu_0 + \frac{1}{2}m_0(t)\cdot\nabla u_{m_0}(t) - h(t)\cdot m_0(t)\Big)\mathrm{d}x$$

$$+ \int_0^t\int_\Omega\Big(\varkappa\big(\nabla\Delta^{-1}(\lambda_\varkappa(s)-L\bullet\nu_\varkappa(s))\big)\cdot\big(\nabla\Delta^{-1}\dot\lambda_\varkappa(s)\big) - \dot h(s)\cdot m(s)\Big)\mathrm{d}x\mathrm{d}s$$

for any $t \in [0,T]$. Combining this with the flow rule tested by $\dot\lambda_\varkappa$ gives that $\sup_{t\in[0,T]}\int_\Omega|\cdot|^p\bullet\nu_\varkappa$ is bounded independently of $\varkappa$ and, moreover, that $\lambda_\varkappa$ is bounded independently of $\varkappa$ in $W^{1,2}([0,T];L^2(\Omega;\mathbb{R}^{d+1}))$, too. Using the estimates for $\lambda_\varkappa$ once again in the flow-rule, we get that $\int_0^T|\varkappa\int_\Omega\nabla\Delta^{-1}(\lambda_\varkappa-L\bullet\nu_\varkappa)\nabla\Delta^{-1}v\mathrm{d}x|\mathrm{d}t$ is bounded for all $v \in L^2(Q;\mathbb{R}^{d+1})$ such that $\|v\|_{L^2(Q;\mathbb{R}^{d+1})} \le 1$, in particular $\{|\varkappa\int_\Omega\nabla\Delta^{-1}(\lambda_\varkappa(t)-L\bullet\nu_\varkappa(t))\nabla\Delta^{-1}v_S\mathrm{d}x|\}_{\varkappa>0}$ is bounded for all $v_S \in L^2(\Omega;\mathbb{R}^{d+1})$ such that $\|v_S\|_{L^2(\Omega;\mathbb{R}^{d+1})} \le 1$ and a.a. $t \in [0,T]$. This in turn means that $\{\varkappa\|\lambda_\varkappa(t) - L\bullet\nu_\varkappa(t)\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\}_{\varkappa>0}$ is bounded for almost all $t \in [0,T]$.

Using that $W^{1,2}([0,T];L^2(\Omega;\mathbb{R}^{d+1})) \subset C([0,T];L^2(\Omega;\mathbb{R}^{d+1}))$ one can select a subsequence of $\varkappa$ (not relabeled) and find $\lambda \in W^{1,2}([0,T];L^2(\Omega;\mathbb{R}^{d+1}))$, such that $\lambda_\varkappa(t) \to \lambda(t)$ strongly in $H^{-1}(\Omega;\mathbb{R}^{d+1})$ for all $t \in [0,T]$.

Let us fix some $t \in [0,T]$ *such that* $\{\varkappa\|\lambda_\varkappa(t) - L\bullet\nu_\varkappa(t)\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\}_{\varkappa>0}$ *is bounded (note that this is possible a.a.* $t \in \Omega$). Then similarly as in the isothermal rate-independent case one can find a subsequence of $\varkappa$ (dependent on t) denoted $\varkappa_t$ and $\nu \in \mathscr{Y}^p(\Omega;\mathbb{R}^d)$ such that $\nu_{\varkappa_t} \to \nu$ weakly* in $\mathscr{Y}^p(\Omega;\mathbb{R}^d)$. Once again by lower semi-continuity one can get the minimization principle

$$\int_\Omega\Big(\phi\bullet\nu\frac{1}{2}m\cdot\nabla u_m-h(t)\cdot m\Big)\mathrm{d}x \le \liminf_{\varkappa_t\to\infty}\int_\Omega\Big(\phi\bullet\nu_{\varkappa_t}+\frac{1}{2}m_{\varkappa_t}\cdot\nabla u_{m_{\varkappa_t}}-h(t)\cdot m_{\varkappa_t}\Big)\mathrm{d}x$$

$$\le \liminf_{\varkappa_t\to\infty}\int_\Omega\Big(\phi\bullet\nu_{\varkappa_t}+\frac{1}{2}m_{\varkappa_t}\cdot\nabla u_{m_{\varkappa_t}}-h(t)\cdot m_{\varkappa_t}\Big)\mathrm{d}x+\frac{\varkappa_t}{2}\|\lambda_{\varkappa_t}-L\bullet\nu_{\varkappa_t}\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$$

$$\le \liminf_{\varkappa_t\to\infty}\int_\Omega\Big(\phi\bullet\hat\nu+\frac{1}{2}\hat m\nabla u_{\hat m}-h(t)\cdot\hat m\Big)\mathrm{d}x+\frac{\varkappa_t}{2}\|\lambda_{\varkappa_t}(t)-L\bullet\hat\nu\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \qquad (2.21)$$

for any $\hat{\nu} \in \mathscr{Y}^p(\Omega; \mathbb{R}^d)$. It would seem logical to take $\hat{\nu}$ such that $\lambda(t) = L \bullet \hat{\nu}$, then the penalization term on the right-hand side would become $\varkappa \|\lambda_\varkappa(t) - \lambda(t)\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$. However, although we know that $\|\lambda_\varkappa(t) - \lambda(t)\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$ converges to 0 as $\varkappa \to \infty$ this no longer needs to hold if the term is multiplied by $\varkappa$.

The limit passage in the minimization principle in the thermally coupled rate-dependent case therefore seems to be much more involved than in the cases presented before. This is mainly due to the fact that now the evolution of λ is given by a completely separate equation. However, we can use a trick to circumpass this problem. Namely we realize that, *due to the convexity*, any solution of (2.13a) $\nu_\varkappa$, solves also the following problem (for the fixed t)

$$
\begin{aligned}
&\text{minimize} \quad \int_\Omega \Big(\phi \bullet \nu + \frac{1}{2} m \cdot \nabla u_m - h(t) \cdot m + \varkappa (\nabla \Delta^{-1}(\lambda_\varkappa(t) - L \bullet \nu_\varkappa)) \cdot (\nabla \Delta^{-1}(\lambda_\varkappa(t) - L \bullet \nu)) \Big) \, \mathrm{d}x \\
&\text{subject to} \quad m = \mathrm{id} \bullet \nu, \quad h_{\mathrm{dem}} = \nabla u_m \quad \nu_\varkappa \text{ solution to (2.13a)} \qquad \text{on } \Omega, \\
&\qquad\qquad \mathrm{div}\big(\mu_0 \nabla u_m - \chi_\Omega m\big) = 0 \qquad \text{on } \mathbb{R}^d, \\
&\qquad\qquad \nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d), \; m \in L^p(\Omega; \mathbb{R}^d), \; u_m \in H^1(\mathbb{R}^d).
\end{aligned}
\tag{2.22}
$$

Now we return to the second line (2.21) (and use that $\varkappa_t/2 \le \varkappa_t$) but instead of exploiting that $\nu_{\varkappa_t}$ solves (2.13a) we use that it solves (2.22) and get

$$
\liminf_{\varkappa_t \to \infty} \int_\Omega \Big(\phi \bullet \nu_{\varkappa_t} + \frac{1}{2} m_{\varkappa_t} \cdot \nabla u_{m_{\varkappa_t}} - h(t) \cdot m_{\varkappa_t} \Big) \, \mathrm{d}x + \varkappa_t \|\lambda_{\varkappa_t} - L \bullet \nu_{\varkappa_t}\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})}
$$

$$
= \liminf_{\varkappa_t \to \infty} \int_\Omega \Big(\phi \bullet \nu_{\varkappa_t} + \frac{1}{2} m_{\varkappa_t} \cdot \nabla u_{m_{\varkappa_t}} - h(t) \cdot m_{\varkappa_t} + \varkappa_t \nabla \Delta^{-1}(\lambda_{\varkappa_t} - L \bullet \nu_{\varkappa_t}) \nabla \Delta^{-1}(\lambda_{\varkappa_t} - L \bullet \nu_{\varkappa_t}) \Big) \mathrm{d}x
$$

$$
\le \liminf_{\varkappa_t \to \infty} \int_\Omega \Big(\phi \bullet \hat{\nu} + \frac{1}{2} \hat{m} \nabla u_{\hat{m}} - h(t) \cdot \hat{m} + \varkappa_t \nabla \Delta^{-1}(\lambda_{\varkappa_t} - L \bullet \nu_{\varkappa_t}) \nabla \Delta^{-1}(\lambda_{\varkappa_t} - L \bullet \hat{\nu}) \Big) \mathrm{d}x
$$

for all $\hat{\nu} \in \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ such that $\lambda = L \bullet \hat{\nu}$. Now we exploit our special choice of t for which we know that $\varkappa_t \|\lambda_{\varkappa_t}(t) - L \bullet \nu_{\varkappa_t}(t)\|_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$ is bounded and $\lambda_{\varkappa_t}(t) \to \lambda(t)$ strongly in $H^{-1}(\Omega; \mathbb{R}^{d+1})$, which makes the penalization term vanish. Hence we establish the converged minimization principle.

3. Weak formulation, qualification and main results

In this section we shall give a weak formulation of the proposed micromagnetics model. The used formulation is to a great extend inspired by the *energetic formulation* for rate-independent processes (see e.g. [26]) and its generalization given in e.g. [34] for problems that include both rate-independent and rate-dependent processes.

In our case as well, we may regard the magnetic variable ν to be fast evolving; its evolution is therefore driven by rate-independent processes. On the other hand, the variables θ and λ are evolving slowly as there evolution is driven by a rate-dependent process. Hence, we demand the fast evolving variable to satisfy a minimization principle and for the slow variables we just require standard weak formulation of (2.13,b-d).

Definition 3.1 (WEAK SOLUTION). *The triple* $(\nu, \lambda, w) \in (\mathscr{Y}^p(\Omega; \mathbb{R}^d))^{[0,T]} \times W^{1,q}([0,T]; L^q(\Omega; \mathbb{R}^{d+1})) \times L^1([0,T]; W^{1,1}(\Omega))$ *such that* $m = \mathrm{id} \bullet \nu \in L^2(Q; \mathbb{R}^d)$ *and* $L \bullet \nu \in L^2(Q; \mathbb{R}^{d+1})$ *is called a weak solution to* (2.13) *if it satisfies:*

1. *The* **minimization principle***: For all* $\tilde{\nu}$ *in* $\mathscr{Y}^p(\Omega; \mathbb{R}^d)$ *and all* $t \in [0, T]$

$$
\mathscr{G}(t, \nu, \lambda, \Theta(w)) \le \mathscr{G}(t, \tilde{\nu}, \lambda, \Theta(w)).
\tag{3.1}
$$

2. *The **reduced Maxwell system for magnetostatics**: For a.a. $t \in [0, T]$ and all $v \in H^1(\mathbb{R}^d)$*

$$\mu_0 \int_{\mathbb{R}^d} \nabla u_m \cdot \nabla v \, \mathrm{d}x = \int_\Omega m \cdot \nabla v \, \mathrm{d}x. \tag{3.2}$$

3. *The **flow rule**: For any $v \in L^q(Q; \mathbb{R}^{d+1})$*

$$\int_Q \left(\left(\Theta(w) - \theta_c\right) \vec{a} \cdot (v - \dot{\lambda}) + \delta_S^*(v) + \frac{\epsilon}{q}|v|^q + \varkappa \nabla \Delta^{-1}(\lambda - L \bullet \nu) \cdot \nabla \Delta^{-1}(v - \dot{\lambda}) \right) \mathrm{d}x\mathrm{d}t$$

$$\geq \int_Q \left(\delta_S^*(\dot{\lambda}) + \frac{\epsilon}{q}|\dot{\lambda}|^q \right) \mathrm{d}x\mathrm{d}t. \tag{3.3}$$

4. *The **enthalpy equation**: For any $\varphi \in C^1(\bar{Q})$, $\varphi(T) = 0$*

$$\int_Q \left(\mathcal{K}(\lambda, w)\nabla w \cdot \nabla \varphi - w\dot{\varphi} \right) \mathrm{d}x\mathrm{d}t + \int_\Sigma b\Theta(w)\varphi \, \mathrm{d}S\mathrm{d}t = \int_\Omega w_0\varphi(0) \, \mathrm{d}x$$

$$+ \int_Q \left(\delta_S^*(\dot{\lambda}) + \epsilon|\dot{\lambda}|^q + \Theta(w)\vec{a}\cdot\dot{\lambda} \right)\varphi \, \mathrm{d}x\mathrm{d}t + \int_\Sigma b\theta_{\mathrm{ext}}\varphi \, \mathrm{d}S\mathrm{d}t. \tag{3.4}$$

5. *The **remaining initial conditions** in (2.14): $\nu(0, \cdot) = \nu_0$ and $\lambda(0, \cdot) = \lambda_0$.*

Data qualifications:
Let us now summarize the data qualification, needed to prove the existence of weak solutions:

Isothermal part of the anisotropy energy: $\phi \in C(\mathbb{R}^d)$ and

$$\exists c_1^A, c_2^A > 0, \ p > 4: \quad c_1^A(1 + |\cdot|^p) \leq \phi(\cdot) \leq c_2^A(1 + |\cdot|^p), \tag{3.5a}$$

Rate-independent dissipation: $\delta_S^* \in C(\mathbb{R}^{d+1})$ positively homogeneous, and

$$\exists c_{1,D}, c_{2,D} > 0: \quad c_{1,D}(|\cdot|) \leq \delta_S^*(\cdot) \leq c_{2,D}(|\cdot|), \tag{3.5b}$$

External magnetic field:

$$h \in C^1([0, T]; L^2(\Omega; \mathbb{R}^d)), \tag{3.5c}$$

Specific heat capacity: $c_v \in C(\mathbb{R})$ and, with q from (2.2),

$$\exists c_{1,\theta}, c_{2,\theta} > 0, \ \omega_1 \geq \omega \geq q', \ c_{1,\theta}(1+\theta)^{\omega-1} \leq c_v(\theta) \leq c_{2,\theta}(1+\theta)^{\omega_1-1}, \tag{3.5d}$$

Heat conduction tensor: $\mathcal{K} \in C(\mathbb{R}^{d+1} \times \mathbb{R}; \mathbb{R}^{d\times d})$ and

$$\exists C_K, \kappa_0 > 0 \ \forall \chi \in \mathbb{R}^d : \mathcal{K}(\cdot, \cdot) \leq C_K, \quad \chi^{\mathrm{T}}\mathcal{K}(\cdot, \cdot)\chi \geq \kappa_0|\chi|^2, \tag{3.5e}$$

External temperature:

$$\theta_{\mathrm{ext}} \in L^1(\Sigma), \quad \theta_{\mathrm{ext}} \geq 0, \quad \text{and} \quad b \in L^\infty(\Sigma), \quad b \geq 0, \tag{3.5f}$$

Initial conditions:

$$\nu_0 \in \mathscr{Y}^p(\Omega; \mathbb{R}^d) \text{ solving } (3.1), \quad \lambda_0 \in L^q(\Omega; \mathbb{R}^{d+1}), \quad w_0 = \widehat{c}_v(\theta_0) \in L^1(\Omega) \text{ with } \theta_0 \geq 0. \tag{3.5g}$$

Note that (2.11) combined with (3.5d) yields for non-negative w

$$w = \widehat{c}_v(\theta) = \int_0^\theta c_v(r)\mathrm{d}r \geq c_{1,\theta} \int_0^\theta (1+r)^{\omega-1}\mathrm{d}r \geq c_{1,\theta}\big((1+\theta)^\omega - 1\big) = c_{1,\theta}\big((1+\Theta(w))^\omega - 1\big). \tag{3.6}$$

The main analytical result we will prove in the following sections is:

Theorem 3.2. *Let* (3.5) *hold. Then at least one weak solution* (ν, λ, w) *to the problem* (2.13) *in accord with Definition 3.1 does exist. Moreover, some of these solutions satisfies also*

$$w \in L^r([0,T]; W^{1,r}(\Omega)) \cap W^{1,1}(I; W^{1,\infty}(\Omega)^*) \qquad with\ 1 \le r < \frac{d+2}{d+1}. \tag{3.7}$$

Scenario of the proof. The above assertion will immediately follow from the Proposition 4.3, which assures convergence of the approximate solutions constructed by semi-implicit time discretization in Section 4 to the weak solutions of (2.13), when proving a-priori estimates in Proposition 4.2 and when realizing that the approximations $w_{0,\tau}$ and $\lambda_{0,\tau}$ of w_0 and λ_0 required in (4.2a) below always exist. The information $\frac{\partial w}{\partial t} \in L^1(I; W^{1,\infty}(\Omega)^*)$ can be obtained from the equation (2.13c,d) itself. $\square$

4. Time discretization, a-priori estimates, and convergence

To prove Theorem 3.2, we proceed in a *constructive manner* that may serve also as a *conceptual numerical algorithm*, at least after a spatial discretization being performed. Namely, we discretize the time with a time-step τ and introduce a minimization problem in every time-step that we shall call the *time-incremental minimization problem*. This problem represents a discrete version of the minimization principle as well as of the flow rule. Also, we apply a semi-implicit method of time-discretization in such a way that it decouples the time incremental minimization problem and the heat equation in any particular time-step k by using the "retarded" enthalpy, i.e. w_τ^{k-1}.

We call the triple $(\nu_\tau^k, \lambda_\tau^k, w_\tau^k) \in \mathscr{Y}^p(\Omega; \mathbb{R}^d) \times L^{2q}(\Omega; \mathbb{R}^{d+1}) \times H^1(\Omega)$ the *discrete weak solution* of (2.13) subject to boundary condition (2.13d) at time-level k, $k = 1 \ldots T/\tau$, if it satisfies:

1. The time-incemental **minimization problem** with given λ_τ^{k-1} and w_τ^{k-1}:

$$\left.\begin{array}{l} \text{Minimize} \quad \mathscr{G}(k\tau, \nu, \lambda, \Theta(w_\tau^{k-1})) + \tau \int_\Omega \left(|\lambda|^{2q} + \delta_S^*\left(\frac{\lambda - \lambda_\tau^{k-1}}{\tau}\right) + \frac{\epsilon}{q}\left|\frac{\lambda - \lambda_\tau^{k-1}}{\tau}\right|^q \right) \mathrm{d}x \\ \text{subject to} \quad (\nu, \lambda) \in \mathscr{Y}^p(\Omega; \mathbb{R}^d) \times L^{2q}(\Omega; \mathbb{R}^{d+1}). \end{array}\right\} \tag{4.1a}$$

   with $\mathscr{G}$ from (2.4).
2. The **reduced Maxwell system for magnetostatics**: For all $v \in H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \nabla u_{m_\tau^k} \cdot \nabla v \, \mathrm{d}x = \int_\Omega m_\tau^k \cdot \nabla v \, \mathrm{d}x \qquad \text{with} \ \ m_\tau^k = \mathrm{id} \bullet \nu_\tau^k. \tag{4.1b}$$

3. The **enthalpy equation**: For all $\varphi \in H^1(\Omega)$

$$\int_\Omega \left(\frac{w_\tau^k - w_\tau^{k-1}}{\tau}\varphi + \mathcal{K}(\lambda_\tau^k, w_\tau^k)\nabla w_\tau^k \cdot \nabla\varphi \right) \mathrm{d}x + \int_\Gamma b_\tau^k \Theta(w_\tau^k)\varphi \, \mathrm{d}S = \int_\Gamma b_\tau^k \theta_{\mathrm{ext},\tau}^k \varphi \, \mathrm{d}S$$
$$+ \int_\Omega \left(\delta_S^*\left(\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau}\right) + \epsilon\left|\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau}\right|^q \Theta(w_\tau^k)\vec{a} \cdot \frac{\lambda^k - \lambda^{k-1}}{\tau} \right) \varphi \, \mathrm{d}x. \tag{4.1c}$$

4. For $k = 0$ the **initial conditions** in the following sense

$$\nu_\tau^0 = \nu_0, \qquad \lambda_\tau^0 = \lambda_{0,\tau}, \qquad w_\tau^0 = w_{0,\tau} \quad \text{on } \Omega. \tag{4.1d}$$

In (4.1d), we denoted by $\lambda_{0,\tau} \in L^{2q}(\Omega; \mathbb{R}^{d+1})$ and $w_{0,\tau} \in L^2(\Omega)$ respectively suitable approximation of the original initial conditions $\lambda_0 \in L^q(\Omega; \mathbb{R}^{d+1})$ and $w_0 \in L^1(\Omega)$ such that

$$\lambda_{0,\tau} \to \lambda_0 \ \text{ strongly in } L^q(\Omega; \mathbb{R}^{d+1}), \text{ and } \ \|\lambda_{0,\tau}\|_{L^{2q}(\Omega; \mathbb{R}^{d+1})} \le C\tau^{-1/(2q+1)}, \tag{4.2a}$$

$$w_{0,\tau} \to w_0 \ \text{ strongly in } L^1(\Omega), \text{ and } \ w_{0,\tau} \in L^2(\Omega). \tag{4.2b}$$

Moreover $\theta_{\text{ext},\tau}^k \in L^2(\Gamma)$ and $b_\tau^k \in L^\infty(\Gamma)$ are defined in such a way that their piecewise constant interpolants

$$[\bar{\theta}_{\text{ext},\tau}, \bar{b}_\tau](t) := (\theta_{\text{ext},\tau}^k, b_\tau^k,) \qquad \text{for} \quad (k-1)\tau < t \le k\tau, \ \ k = 1, ..., K_\tau.$$

satisfy

$$\bar{\theta}_{\text{ext},\tau} \to \theta_{\text{ext}} \ \text{ strongly in } L^1(\Sigma) \text{ and } \ \bar{b}_\tau \overset{*}{\rightharpoonup} b \ \text{ weakly* in } L^\infty(\Sigma). \tag{4.3}$$

To see why (4.1) indeed forms a correct time-discretization of the weak formulation of (3.1), note that (4.1a) already contains the flow rule since, if we can find a minimizer, the first order optimality condition in λ evaluated at λ_τ^k yields the discrete version of (3.3).

Realize also, that we have added the regularization term $\tau|\lambda|^{2q}$ to the time-incremental problem, cf. (4.1a). This shall assure that even $|\frac{\lambda^k - \lambda^{k-1}}{\tau}|^q \in L^2(\Omega)$ (although it does not hold uniformly for $\tau \to 0$) and in turn also the existence of solutions of the enthalpy equation in the classical weak sense. Of course, this regularization term will be shown to converge to 0 as we pass to the limit $\tau \to 0$.

Proposition 4.1 (Existence of discrete solutions). *Let* (3.5) *hold and let also* $w_\tau^0 \ge 0$. *Then there exists a discrete weak solution according to* (4.1) *such that* $w_\tau^k \ge 0$ *for all* $k = 1 \ldots T/\tau$.

Proof. First note that (4.1c) is decoupled from (4.1a) and (4.1b). It is easy to see that, for any $\nu \in \mathscr{Y}^p(\Omega; \mathbb{R}^d)$, there exists a unique $u_m \in H^1(\mathbb{R}^d)$ that solves (4.1b) since $m \equiv \text{id} \bullet \nu \in L^2(\Omega; \mathbb{R}^d)$. Therefore we may proceed by the *direct method* to prove existence of (4.1a), (4.1b), i.e. take at time-step k a minimization sequence $\{q_{k,j}\}_{j=0}^\infty = \{(\nu_{k,j}, \lambda_{k,j})\}_{j=0}^\infty$ of (4.1a). Due to the coercivity of the cost functional (thanks to assumption (3.5a)) this minimizing sequence converges weakly*(at least in terms of a subsequence) in the space $L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^d)) \times L^q(\Omega; \mathbb{R}^{d+1})$ to some q_k. Moreover, note that, again due to assumption (3.5a), ν_k is a Young measure. Then, by the convexity of the functional in λ, ν, by the fact that $m_{k,j} \rightharpoonup m_k$ in $L^2(\Omega; \mathbb{R}^d)$ (because $\nu_{k,j}$ as well as ν_k are in $\mathscr{Y}^p(\Omega; \mathbb{R}^d)$ and $\nu_{k,j} \overset{*}{\rightharpoonup} \nu_k$ in $L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^d))$) and the boundedness from below of ϕ, q_k is the sought minimizer of (4.1a) at time-step k.

The existence of solutions to (4.1c) for $k = 1$ (and subsequently also for all other k) can be proved by standard methods exploiting theory of pseudomonotone operators (here rather trivially since the problem is semi-linear and thus the underlying operator $H^1(\Omega) \to H^1(\Omega)^*$ is weakly continuous). Note that the right-hand side can be represented as an element of $H^1(\Omega)^*$ due to the integrability of the initial data, the suitable choice of the time-discretization of the external heat flux $\theta_{\text{ext},\tau}^k \in L^2(\Gamma)$, and thanks to the regularization term $\tau|\lambda|^{2q}$ in (4.1a) which causes $|\frac{\lambda^k - \lambda^{k-1}}{\tau}|^q \in L^2(\Omega)$, as already mentioned above.

Let us test (4.1c) by $[w_\tau^k]^- \equiv \min(0, w_\tau^k)$, which is an element of $H^1(\Omega)$ as $w_\tau^k \in H^1(\Omega)$ and hence a legal test function. We get

$$\int_\Omega \left(w_\tau^k [w_\tau^k]^- + \tau \mathcal{K}(\lambda_\tau^k, w_\tau^k) \nabla w_\tau^k \cdot \nabla [w_\tau^k]^- \right) dx \le \int_\Omega \left(\delta_S^* \left(\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau} \right) [w_\tau^k]^- + \epsilon \left| \frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau} \right|^q [w_\tau^k]^- \right.$$
$$\left. + \Theta(w_\tau) \vec{a} \cdot \frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau} [w_\tau^k]^- + w_\tau^{k-1} [w_\tau^k]^- \right) dx - \int_\Gamma \left(b_\tau^k \Theta(w_\tau^k)[w_\tau^k]^- + b_\tau^k \theta_{\text{ext},\tau}^k [w_\tau^k]^- \right) dS.$$

Now $\Theta(w_\tau^k)[w_\tau^k]^- = 0$; here we realize that we have defined $\Theta(w) = 0$ for $w \le 0$. Further we realize that $\delta_S^*(\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau}) \ge 0$ and $\epsilon|\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau}|^q \ge 0$ which implies that $\delta_S^*(\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau})[w_\tau^k]^- \le 0$ and $\epsilon|\frac{\lambda_\tau^k - \lambda_\tau^{k-1}}{\tau}|^q [w_\tau^k]^- \le 0$. Using these facts and by exploiting that $b_\tau^k \theta_{\text{ext},\tau}[w_\tau^k]^- \le 0$ (a consequence of (3.5f)), we get

$$\int_\Omega \left| [w_\tau^k]^- \right|^2 + \tau \kappa_0 \left| \nabla [w_\tau^k]^- \right|^2 dx \le \int_\Omega w_\tau^{k-1} [w_\tau^k]^- dx.$$

Using this equation recursively and when also taking into account that $w_\tau^0 \ge 0$ gives $w_\tau^k \ge 0$. $\quad\square$

Let us introduce the notion of *piecewise affine* interpolants λ_τ and w_τ defined by

$$\left[\lambda_\tau, w_\tau\right](t) := \frac{t-(k-1)\tau}{\tau}\left(\lambda_\tau^k, w_\tau^k\right) + \frac{k\tau-t}{\tau}\left(\lambda_\tau^{k-1}, w_\tau^{k-1}\right) \qquad \text{for} \ \ t \in [(k-1)\tau, k\tau]$$

with $k = 1, ..., T/\tau$. In addition define the backward *piecewise constant interpolants* $\bar{\nu}_\tau$, $\bar{\lambda}_\tau$, and $\bar{w}_\tau$ by

$$\left[\bar{\nu}_\tau, \bar{\lambda}_\tau, \bar{w}_\tau\right](t) := \left(\nu_\tau^k, \lambda_\tau^k, w_\tau^k\right) \qquad \text{for} \ \ (k-1)\tau < t \le k\tau, \ \ k = 1, ..., T/\tau. \tag{4.4}$$

Eventually, we will also need the "retarded" enthalpy and magnetization piecewise constant interpolant $\underline{w}_\tau$, $\underline{m}_\tau$ defined by

$$[\underline{w}_\tau(t), \underline{m}_\tau(t)] := [w_\tau^{k-1}, \text{id} \bullet \nu_\tau^{k-1}] \qquad \text{for} \ \ (k-1)\tau < t \le k\tau, \ \ k = 1, ..., T/\tau. \tag{4.5}$$

Note that from now on *we use C as a generic constant*, which ma change from expression to expression, and do not specify its dependence on the problem parameters such as $\epsilon, q, p, |\Omega|$.

Proposition 4.2 (A-priori estimates). *Let the assumptions* (3.5) *hold and let $\tau < \tau_0$ for some $\tau_0 > 0$ fixed. Then the interpolants of discrete weak solutions satisfy*

$$\sup_{t\in[0,T]} \int_\Omega |\cdot|^p \bullet \bar{\nu}_\tau \mathrm{d}x \le C, \tag{4.6}$$

$$\left\|\dot{\lambda}_\tau\right\|_{L^q(Q;\mathbb{R}^{d+1})} \le C, \tag{4.7}$$

$$\left\|\bar{\lambda}_\tau\right\|_{L^\infty([0,T];L^{2q}(\Omega;\mathbb{R}^{d+1}))} \le C\tau^{-1/2q}, \tag{4.8}$$

$$\left\|\bar{w}_\tau\right\|_{L^\infty([0,T];L^1(\Omega))} \le C, \tag{4.9}$$

$$\left\|\nabla\bar{w}_\tau\right\|_{L^r(Q;\mathbb{R}^d)} \le C_r \qquad \text{with any} \ \ 1 \le r < \frac{d+2}{d+1}, \tag{4.10}$$

$$\left\|\dot{w}_\tau\right\|_{\mathcal{M}([0,T];W^{1,\infty}(\Omega)^*)} \le C. \tag{4.11}$$

Let us emphasize that our strategy of proving (4.7) and (4.9) will slightly deviate from the standard approach based on testing the flow rule (2.10b) by $\dot{\lambda}$ together with the entropy equation (2.6) by 1, which, when adding these two, would lead to canceling of the dissipative heat rate as well as the calorimetric term $\Theta(w)\vec{a}\cdot\dot{\lambda}$, see e.g. [34]. In contrast to [34] however, here we use the retarded enthalpy in the time-incremental minimization problem and hence the strategy of [34] would not work in our case, as the calorimetric term *would not cancel out*. Therefore we exploit the rate-dependent dissipation term that yields more regularity than the rate-independent contribution.

In what follows, we will use the abbreviation $\langle\langle\cdot,\cdot\rangle\rangle$ for the scalar product in $H^{-1}(\Omega;\mathbb{R}^{d+1})$; in view of the specific choice of the norm on $H^{-1}(\Omega;\mathbb{R}^{d+1})$ in Sect. 2, this means

$$\langle\langle\lambda,v\rangle\rangle := \int_\Omega \nabla\Delta^{-1}\lambda\cdot\nabla\Delta^{-1}v \, \mathrm{d}x. \tag{4.12}$$

Proof of Proposition 4.2. Before giving a rigorous proof we give a formal heuristic sketch, using the system (2.13), on how estimates (4.6)-(4.11) can be established; we shall always point to the adequate step in this proof where the formal procedure is performed rigorously on the discrete level. For this heuristics only, we shall assume that all functions are as smooth as needed. Furthermore, let us denote by (ν, λ, θ) the solutions of (2.13).

First of all we exploit (2.13a), which can be reformulated as minimizing part with respect to ν the magnetic of the Gibbs free energy $\mathfrak{G}$ defined as

$$\mathfrak{G}(t,\nu,\lambda) := \int_\Omega \phi\bullet\nu - h(t)\cdot m \, \mathrm{d}x + \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u_m|^2 \, \mathrm{d}x + \frac{\varkappa}{2}\left\|\lambda - L\bullet\nu\right\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}^2. \tag{4.13}$$

Then just using the chain rule and realizing that the partial derivative of $\mathfrak{G}$ with respect to ν evaluated at the minimizer has to be zero leads to

$$\mathfrak{G}(t_0, \nu(t_0), \lambda(t_0)) = \mathfrak{G}(t_0, \nu(0), \lambda(0)) + \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{G}(t, \nu(t), \lambda(t)) \mathrm{d}t$$

$$= \mathfrak{G}(t_0, \nu(0), \lambda(0)) + \int_0^{t_0} \left(\dot{h}(t) m + \varkappa \langle\!\langle \lambda - L \bullet \nu, \dot{\lambda} \rangle\!\rangle \right) \mathrm{d}t, \qquad (4.14)$$

for somee arbitrary $t_0 \in [0, T]$. Using the Young inequality and assumption (3.5a) on the coercivity of the isothermal part of the anisotropy energy and also (3.5c) yields (cf. Step 1 below) the following estimate

$$C \int_\Omega |\cdot|^p \bullet \nu(t_0) \le \int_0^{t_0} \int_\Omega \left(\frac{\varepsilon}{4q} |\dot{\lambda}|^q + C |\cdot|^p \bullet \nu \right) \mathrm{d}x \mathrm{d}t + C. \qquad (4.15)$$

Inequality (4.15) needs to be combined with an estimate on $\dot{\lambda}$. In order to get it, we multiply the flow-rule (2.13ab) by $\frac{1}{q} \dot{\lambda}$ and integrate over Ω and $[0, t_0]$, which yields (note that $\partial \delta_{\mathrm{S}}^*(\dot{\lambda}) \dot{\lambda} = \delta_{\mathrm{S}}^*(\dot{\lambda})$)

$$\int_0^{t_0} \int_\Omega \frac{1}{q} \left(\delta_{\mathrm{S}}^*(\dot{\lambda}) + \frac{\epsilon}{q} |\dot{\lambda}|^q \right) \mathrm{d}x \mathrm{d}t = \int_0^{t_0} \left(\frac{\varkappa}{q} \langle\!\langle \lambda - L \bullet \nu, \dot{\lambda} \rangle\!\rangle - \frac{1}{q} \int_\Omega (\Theta(w) - \theta_c) \vec{a} \cdot \dot{\lambda} \, \mathrm{d}x \right) \mathrm{d}t.$$

Using the chain-rule, we can rewrite this as

$$\int_0^{t_0} \int_\Omega \frac{1}{q} \left(\delta_{\mathrm{S}}^*(\dot{\lambda}) + \frac{\epsilon}{q} |\dot{\lambda}|^q \right) \mathrm{d}x \mathrm{d}t + \frac{\varkappa}{q} \| \dot{\lambda}(t_0) \|_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$$

$$\le \frac{\varkappa}{q} \| \dot{\lambda}(0) \|_{H^{-1}(\Omega; \mathbb{R}^{d+1})} - \int_0^{t_0} \left(\frac{\varkappa}{q} \langle\!\langle L \bullet \nu, \dot{\lambda} \rangle\!\rangle + \frac{1}{q} \int_\Omega (\Theta(w) - \theta_c) \vec{a} \cdot \dot{\lambda} \, \mathrm{d}x \right) \mathrm{d}t, \qquad (4.16)$$

which, by usage of the Young inequality (cf. Step 2 below), yields the following estimate

$$\int_0^{t_0} \int_\Omega \frac{1}{q} \left(\delta_{\mathrm{S}}^*(\dot{\lambda}) + \frac{\epsilon}{q} |\dot{\lambda}|^q \right) \mathrm{d}x \mathrm{d}t + \frac{\varkappa}{q} \| \dot{\lambda}(t_0) \|_{H^{-1}(\Omega; \mathbb{R}^{d+1})} \le \int_0^{t_0} \int_\Omega \left(\frac{\varepsilon}{4q} |\dot{\lambda}|^q + C |\cdot|^p \bullet \nu + C |w| \right) \mathrm{d}x \mathrm{d}t + C.$$

$$(4.17)$$

Finally, multiplying the enthalpy equation (2.13c) by $\frac{\varepsilon}{8q}$, integrating over Ω and $[0, t_0]$, and using the Young inequality as already above gives

$$\frac{\varepsilon}{8q} \int_0^{t_0} \int_\Omega \dot{w} \mathrm{d}x \mathrm{d}t \le \int_0^{t_0} \int_\Omega \left(\frac{\varepsilon}{4q} |\dot{\lambda}|^q + C |w| \right) \mathrm{d}x \mathrm{d}t + C \qquad (4.18)$$

Adding (4.15), (4.17) and (4.18) gives, on this heuristic level, (4.6), (4.7) and (4.9).

Estimate (4.10) is got by testing the enthalpy equation (2.13c) by $1 - \frac{1}{(1+w)^a}$ (cf. Step 4, below) while (4.11) is got from the enthalpy equation itself.

For clarity, let us divide the formal part of the proof into five steps.

Step 1: Using the time-incremental minimization problem: In this step we perform the procedure that on the heuristic level led to (4.15); together with Step 2 and Step 3 it will give (4.6)–(4.9).

As we know that $(\nu_\tau^l, \lambda_\tau^l)$ solves the time-incremental problem (4.1a), we may write

$$\mathscr{G}(l\tau, \nu_\tau^l, \lambda_\tau^l, \Theta(w_\tau^{l-1})) + \int_\Omega \left(\tau |\lambda_\tau^l|^{2q} + \tau \delta_{S}^* \left(\frac{\lambda_\tau^l - \lambda_\tau^{l-1}}{\tau} \right) + \frac{\tau \epsilon}{q} \left| \frac{\lambda_\tau^l - \lambda_\tau^{l-1}}{\tau} \right|^q \right) \mathrm{d}x$$

$$\le \mathscr{G}(l\tau, \nu_\tau^{l-1}, \lambda_\tau^l, \Theta(w_\tau^{l-1})) + \int_\Omega \left(\tau |\lambda_\tau^l|^{2q} + \tau \delta_{S}^* \left(\frac{\lambda_\tau^l - \lambda_\tau^{l-1}}{\tau} \right) + \frac{\tau \epsilon}{q} \left| \frac{\lambda_\tau^l - \lambda_\tau^{l-1}}{\tau} \right|^q \right) \mathrm{d}x,$$

which can be rewritten using the magnetic part of the Gibbs free energy as

$$\mathfrak{G}(l\tau, \nu_\tau^l, \lambda_\tau^l) \leq \mathfrak{G}(l\tau, \nu_\tau^{l-1}, \lambda_\tau^l)$$

$$\leq \mathfrak{G}((l-1)\tau, \nu_\tau^{l-1}, \lambda_\tau^{l-1}) + \int_{(l-1)\tau}^{l\tau} \left(\mathfrak{G}_t'(l\tau, \nu_\tau^l) + \mathfrak{G}_\lambda'(\nu_\tau^l, \lambda_\tau^l) \right) \mathrm{d}t$$

where the inequality on the second line is got by the discrete chain rule (relying on convexity). Realizing that $\mathfrak{G}_t'(l\tau, \nu_\tau^l) = \int_\Omega \dot{h}_\tau(l\tau) \cdot m_\tau^l \mathrm{d}x$ (h_τ denotes the piece-wise linear approximation of h) and $\mathfrak{G}_\lambda'(\nu_\tau^l, \lambda_\tau^l) = \varkappa \langle\!\langle \lambda_\tau^l - L \bullet \nu_\tau^l, \frac{\lambda_\tau^l - \lambda_\tau^{l-1}}{\tau} \rangle\!\rangle$ and summing from 0 to k gives

$$\mathfrak{G}(t_k, \bar{\nu}_\tau(t_k), \bar{\lambda}_\tau(t_k)) \leq \mathfrak{G}(0, \bar{\nu}_\tau(0), \bar{\lambda}_\tau(0)) + \int_0^{t_k} \left(\int_\Omega \dot{h}_\tau \cdot \bar{m}_\tau \mathrm{d}x + \varkappa \langle\!\langle \bar{\lambda}_\tau - L \bullet \bar{\nu}_\tau, \dot{\lambda}_\tau \rangle\!\rangle \right) \mathrm{d}t \quad (4.19)$$

with $t_k = k\tau$, a discrete analogy of (4.14). Exploiting ones again the discrete chain rule as

$$\frac{1}{2}\|\bar{\lambda}_\tau(0)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})} - \frac{1}{2}\|\bar{\lambda}_\tau(t_k)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\mathrm{d}x \geq -\int_0^{t_k} \langle\!\langle \dot{\lambda}_\tau, \bar{\lambda}_\tau \rangle\!\rangle \mathrm{d}t,$$

we can rewrite (4.19) as

$$\mathfrak{G}(t_k, \bar{\nu}_\tau(t_k), \bar{\lambda}_\tau(t_k)) + \frac{\varkappa}{2}\|\bar{\lambda}_\tau(t_k)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \leq \frac{\varkappa}{2}\|\bar{\lambda}_\tau(0)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$$

$$+ \mathfrak{G}(0, \bar{\nu}_\tau(0), \bar{\lambda}_\tau(0)) + \int_0^{t_k} \left(\int_\Omega \dot{h}_\tau \cdot \bar{m}_\tau \mathrm{d}x - \varkappa \langle\!\langle L \bullet \bar{\nu}_\tau, \dot{\lambda}_\tau \rangle\!\rangle \right) \mathrm{d}t$$

Estimate on the right hand side (using Young inequality and (3.5c))

$$\left| \int_0^{t_k}\int_\Omega \dot{h}_\tau \cdot \bar{m}_\tau \mathrm{d}x\mathrm{d}t \right| \leq C \int_0^{t_k}\int_\Omega \left(|\dot{h}_\tau|^{p'} + |\cdot|^p \bullet \bar{\nu}_\tau \right)\mathrm{d}x\mathrm{d}t \leq C\left(1 + \int_0^{t_k}\int_\Omega |\cdot|^p \bullet \bar{\nu}_\tau \mathrm{d}x\mathrm{d}t\right)$$

$$\left| \int_0^{t_k} \langle\!\langle L \bullet \bar{\nu}_\tau, \dot{\lambda}_\tau \rangle\!\rangle \mathrm{d}t \right| \leq \int_0^{t_k} \varkappa \|L \bullet \bar{\nu}_\tau\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \|\dot{\lambda}_\tau\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\mathrm{d}t$$

$$\leq \int_0^{t_k} \left(\frac{\varepsilon}{4qc_{\mathrm{em}}}\|\dot{\lambda}_\tau\|^q_{H^{-1}(\Omega;\mathbb{R}^{d+1})} + C\|L \bullet \bar{\nu}_\tau\|^{q'}_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \right)\mathrm{d}t$$

$$\leq \int_0^{t_k} \left(\frac{\epsilon}{4q}\|\dot{\lambda}_\tau\|^q_{L^q(\Omega;\mathbb{R}^{d+1})} + C\int_\Omega |\cdot|^p \bullet \bar{\nu}_\tau \mathrm{d}x \right)\mathrm{d}t + C$$

with $p' = \frac{p}{p-1}$ and c_{em} the specific constant for which $\|\dot{\lambda}_\tau\|^q_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \leq c_{\mathrm{em}}\|\dot{\lambda}_\tau\|^q_{L^q(\Omega;\mathbb{R}^{d+1})}$. Note that we also estimated, thanks to $q \geq 2$, $\|L \bullet \bar{\nu}_\tau\|^{q'}_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \leq C(1+\|L \bullet \bar{\nu}_\tau\|^2_{L^2(\Omega;\mathbb{R}^{d+1})}) \leq C(1 + \int_\Omega |L|^2 \bullet \bar{\nu}_\tau \mathrm{d}x) \leq C(1 + \int_\Omega |\cdot|^p \bullet \bar{\nu}_\tau \mathrm{d}x)$.

On the other hand, we may estimate $\mathfrak{G}(t_k, \bar{\nu}_\tau(t_k), \bar{\lambda}_\tau(t_k))$ from above by using (3.5a) as

$$\mathfrak{G}(t_k, \bar{\nu}_\tau(t_k), \bar{\lambda}_\tau(t_k)) \geq \int_\Omega \phi \bullet \bar{\nu}_\tau(t_k)\mathrm{d}x - \left| \int_\Omega h(t_k) \cdot \bar{m}_\tau(t_k)\mathrm{d}x \right|$$

$$\geq \int_\Omega c_1^A\left(1 + |\cdot|^p \bullet \bar{\nu}_\tau(t_k)\right)\mathrm{d}x - \int_\Omega \left(C|h|^{p'} + \frac{c_1^A}{2}|\cdot|^p \bullet \bar{\nu}_\tau \right)\mathrm{d}x$$

$$\geq C\left(\int_\Omega |\cdot|^p \bullet \bar{\nu}_\tau(t_k)\mathrm{d}x - 1 \right) \quad (4.20)$$

Combining (4.20) with the estimates on the right-hand side of (4.19) one gets

$$C\int_\Omega |\cdot|^p \bullet \bar{\nu}_\tau(t_k) + \frac{\varkappa}{2}\int_\Omega |\bar{\lambda}_\tau(t_k)|^{2q}\mathrm{d}x \leq \int_0^{t_k}\int_\Omega \left(\frac{\varepsilon}{4q}|\dot{\lambda}_\tau|^q + C|\cdot|^p \bullet \bar{\nu}_\tau \right)\mathrm{d}x\mathrm{d}t + C, \quad (4.21)$$

a discrete analogy of (4.15).

Step 2: Testing the flow-rule by $\dot{\lambda}_\tau$: In this step we perform the procedure that on the heuristic level lead to (4.17); together with Step 1 and Step 3 it will give (4.6)–(4.9).

First note that, as $(\nu_\tau^l, \lambda_\tau^l)$ is a minimizer of (4.1a), the partial sub-differential of the cost functional with respect to λ has to be zero at λ_τ^l. Realizing that this condition holds at each time level and summing these conditions up to some k leads to

$$\int_0^{t_k} \int_\Omega \left(\delta_S^*(\dot{\lambda}_\tau) + \frac{\epsilon}{q}|\dot{\lambda}_\tau|^q \right) \mathrm{d}x\mathrm{d}t \leq \int_0^{t_k} \left(\varkappa \langle\!\langle \bar{\lambda}_\tau - L \bullet \bar{\nu}_\tau, v_\tau - \dot{\lambda}_\tau \rangle\!\rangle \right.$$
$$\left. + \int_\Omega \left((\Theta(\underline{w}_\tau) - \theta_\mathrm{c})\vec{a}\cdot(v_\tau - \dot{\lambda}_\tau) + 2q\tau|\bar{\lambda}_\tau|^{2q-2}\bar{\lambda}_\tau(v_\tau - \dot{\lambda}_\tau) + \delta_S^*(v_\tau) + \frac{\epsilon}{q}|v_\tau|^q \right) \mathrm{d}x \right) \mathrm{d}t, \quad (4.22)$$

where v_τ is an arbitrary test function such that $v_\tau(\cdot, x)$ is piecewise constant on the intervals $(t_{j-1}, t_j]$ and $v_\tau(t_j, \cdot) \in L^{2q}(\Omega; \mathbb{R}^{d+1})$ for every j. As

$$\frac{1}{2}\|\bar{\lambda}_\tau(0)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})} - \frac{1}{2}\|\bar{\lambda}_\tau(t_k)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\mathrm{d}x \geq -\int_0^{t_k} \langle\!\langle \dot{\lambda}_\tau, \bar{\lambda}_\tau \rangle\!\rangle \mathrm{d}t \quad \text{and}$$

$$\int_\Omega |\bar{\lambda}_\tau(0)|^{2q} - \int_\Omega |\bar{\lambda}_\tau(t_k)|^{2q}\mathrm{d}x \geq -2q\int_0^{t_k} \int_\Omega \dot{\lambda}_\tau\bar{\lambda}_\tau|\bar{\lambda}_\tau|^{2q-2}\mathrm{d}x\mathrm{d}t,$$

hold by the discrete chain rule (thanks to the convexity of the involved functions on the left-hand-side), we may test (4.22) by $v_\tau = 0$, which effectively executes the test of the discrete version of the inclusion (2.10b) by $\frac{1}{q}\dot{\lambda}_\tau$. This yields

$$\int_0^{t_k} \int_\Omega \left(\delta_S^*(\dot{\lambda}_\tau) + \frac{\epsilon}{q}|\dot{\lambda}_\tau|^q \right) \mathrm{d}x\mathrm{d}t + \frac{\varkappa}{2}\|\bar{\lambda}_\tau(t_k)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})} + \int_\Omega \tau|\bar{\lambda}_\tau(t_k)|^{2q}\mathrm{d}x$$

$$\leq \int_0^{t_k} \left(\varkappa\|L \bullet \bar{\nu}_\tau\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\|\dot{\lambda}_\tau\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})} + \int_\Omega \left(|\Theta(\underline{w}_\tau) - \theta_\mathrm{c}|\,|\vec{a}|\,|\dot{\lambda}_\tau| \right)\mathrm{d}x \right)\mathrm{d}t$$

$$+ \frac{\varkappa}{2}\|\bar{\lambda}_\tau(0)\|^2_{H^{-1}(\Omega;\mathbb{R}^{d+1})} + \int_\Omega \tau|\bar{\lambda}_\tau(0)|^{2q}\mathrm{d}x$$

$$\leq \int_0^{t_k} \left(C\|L \bullet \bar{\nu}_\tau\|^{q'}_{H^{-1}(\Omega;\mathbb{R}^{d+1})} + \frac{\epsilon}{8qc_\mathrm{em}}\|\dot{\lambda}_\tau\|^q_{H^{-1}(\Omega;\mathbb{R}^{d+1})} \right.$$

$$\left. + \int_\Omega \left(C|\Theta(\underline{w}_\tau) - \theta_\mathrm{c}|^{q'} + \frac{\epsilon}{8q}|\dot{\lambda}_\tau|^q \right)\mathrm{d}x \right)\mathrm{d}t + C$$

$$\leq \int_0^{t_k} \int_\Omega \left(C|\cdot|^p \bullet \nu_\tau + C|\bar{w}_\tau| + \frac{\epsilon}{4q}|\dot{\lambda}_\tau|^q \right)\mathrm{d}x\mathrm{d}t + \tau\|\lambda_{0,\tau}\|^{2q}_{L^{2q}(\Omega;\mathbb{R}^{d+1})} + C, \quad (4.23)$$

by applying Young's inequality to the terms $\varkappa\|L \bullet \bar{\nu}_\tau\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}\|\dot{\lambda}_\tau\|_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ as well as $|\Theta(\underline{w}_\tau) - \theta_\mathrm{c}|\,|\vec{a}|\,|\dot{\lambda}_\tau|$. Subsequently we estimated $\|L \bullet \bar{\nu}_\tau\|^{q'}_{H^{-1}(\Omega;\mathbb{R}^{d+1})}$ the same way ay in Step 1; c_em was also chosen like in Step 1. Eventually, $|\Theta(\underline{w}_\tau)|^{q'} \leq C(1 + |\underline{w}_\tau|^{q'/\omega}) \leq C(1 + |\underline{w}_\tau|)$ due to (3.6) and $q'/\omega < 1$ (cf. (3.5d)).

Step 3: Testing the heat equation by 1: In this step we perform the procedure that on the heuristic level lead to (4.17); together with Step 1 and Step 3 it will give (4.6)–(4.9). Summing the discrete version of the enthalpy equation (4.1c) from 0 to t_k leads to

$$\int_0^{t_k} \left(\int_\Omega \left(\dot{w}_\tau\varphi + \mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau)\nabla\bar{w}_\tau\cdot\nabla\varphi \right)\mathrm{d}x + \int_\Gamma \bar{b}_\tau\Theta(\bar{w}_\tau)\varphi\mathrm{d}S \right)\mathrm{d}t$$

$$= \int_0^{t_k} \left(\int_\Omega \left(\delta_S^*(\dot{\lambda}_\tau) + \epsilon|\dot{\lambda}_\tau|^q + \Theta(\bar{w}_\tau)\vec{a}\cdot\dot{\lambda}_\tau \right)\varphi\mathrm{d}x + \int_\Gamma \bar{b}_\tau\bar{\theta}_{\mathrm{ext},\tau}\varphi\mathrm{d}S \right)\mathrm{d}t, \quad (4.24)$$

where φ is an arbitrary test function, such that $\varphi(\cdot, x)$ is piecewise constant on the intervals $(t_{j-1}, t_j]$ and $\varphi(t_j, \cdot) \in H^1(\Omega)$ for every j. Then we test equation (4.24) by 1 to get

$$\int_0^{t_k} \left(\int_\Omega \dot{w}_\tau dx + \int_\Gamma \bar{b}_\tau \Theta(\bar{w}_\tau) dS \right) dt \leq \int_0^{t_k} \left(\int_\Omega \left(\delta_S^*(\dot{\lambda}_\tau) + \epsilon|\dot{\lambda}_\tau|^q + |\Theta(\bar{w}_\tau)| |\vec{a}| |\dot{\lambda}_\tau| \right) dx \right.$$
$$\left. + \int_\Gamma |\bar{b}_\tau \bar{\theta}_{\text{ext},\tau}| dS \right) dt.$$

Estimate the third term on the right-hand side similarly as in Step 2 to arrive at the expression

$$\int_0^{t_k} \int_\Omega \dot{w}_\tau(t, x) dx + \int_\Gamma \bar{b}_\tau \Theta(\bar{w}_\tau) dS dt \leq \int_Q 2|\dot{\lambda}_\tau|^q + C|\bar{w}_\tau| dx dt.$$

Multiplying this by $\varepsilon/(8q)$ and adding to (4.23) and (4.21) already yields (4.6)–(4.9) by the usage of the discrete Gronwall inequality.

Step 4: Estimation of $\nabla \bar{w}_\tau$: In this step we prove (4.10).

Let us test (4.24) by $\eta(\bar{w}_\tau)$ where $\eta(w) = 1 - \frac{1}{(1+w)^a}$ with $a > 0$, which, due to the non-negativity of the enthalpy is a legal test. Notice that due to the discrete chain rule relying on the convexity of $\tilde{\eta}$

$$\int_\Omega \tilde{\eta}(T) - \tilde{\eta}(0) dx = \int_Q \frac{d}{dt} \tilde{\eta}(\bar{w}_\tau) dx dt \leq \int_Q \dot{w}_\tau \eta(\bar{w}_\tau) dx dt,$$

where $\tilde{\eta}$ denotes the primitive function of η such that $\tilde{\eta}(0) = 0$. Realize also that due to the fact that $\eta(w) \geq 0$ also $\tilde{\eta}(T) \geq 0$ and hence we may write

$$\kappa_0 a \int_Q \frac{|\nabla \bar{w}_\tau|^2}{(1+\bar{w})^{1+a}} dx dt = \kappa_0 \int_Q |\nabla \bar{w}_\tau|^2 \eta'(\bar{w}_\tau) dx dt \leq \int_Q \mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau) \nabla \bar{w}_\tau \cdot \nabla \bar{w}_\tau \eta'(\bar{w}_\tau) dx dt$$
$$= \int_Q \mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau) \nabla \bar{w}_\tau \cdot \nabla \eta(\bar{w}_\tau) dx dt$$
$$\leq \int_\Omega \tilde{\eta}(w_\tau(T)) dx + \int_Q \mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau) \nabla \bar{w}_\tau \cdot \nabla \eta(\bar{w}_\tau) dx dt + \int_\Sigma \bar{b}_\tau \Theta(\bar{w}_\tau) \eta(\bar{w}_\tau) dS dt$$
$$= \int_\Sigma \bar{b}_\tau \bar{\theta}_{\text{ext},\tau} \eta(\bar{w}_\tau) dS dt + \int_\Omega \tilde{\eta}(w_0) dx + \int_Q (\delta_S^*(\dot{\lambda}_\tau) + \Theta(\bar{w}_\tau) \vec{a} \cdot \dot{\lambda}_\tau + \epsilon|\dot{\lambda}_\tau|^q) \eta(\bar{w}_\tau) dx dt$$
$$\leq C + \|\bar{r}_\tau\|_{L^1(Q)} \tag{4.25}$$

where we used the obvious bound $|\eta(\bar{w}_\tau)| \leq 1$ and abbreviated

$$\bar{r}_\tau := \delta_S^*(\dot{\lambda}_\tau) + \epsilon|\dot{\lambda}_\tau|^q + \Theta(\bar{w}_\tau) \vec{a} \cdot \dot{\lambda}_\tau.$$

As to the L^1-bound of $\bar{r}_\tau$, realize that

$$\int_Q \delta_S^*(\dot{\lambda}_\tau) + \epsilon|\dot{\lambda}_\tau|^q dx dt \leq 2\epsilon\|\dot{\lambda}_\tau\|_{L^q(Q)}^q + C \quad \text{and}$$

$$\int_Q \left| \Theta(\bar{w}_\tau) \vec{a} \cdot \dot{\lambda}_\tau \right| dx dt \leq C \left(1 + \|\bar{w}_\tau(t)\|_{L^1(Q)} + \epsilon\|\dot{\lambda}_\tau\|_{L^q(Q)}^q \right),$$

similarly as in Step 1 or Step 2 of this proof, and use the already shown estimates (4.7), (4.9). Thus, (4.25) yields $\int_Q \frac{|\nabla \bar{w}_\tau|^2}{(1+\bar{w})^{1+a}} dx dt$ bounded. Combining it with (4.9) like in [3, 4], cf. also [34, Formulae (4.29)-(4.33)], we obtain (4.10).

Step 5: "Dual" estimate for the time derivative: Notice that

$$\|\dot{\bar{w}}_\tau\|_{\mathcal{M}([0,T],W^{1,\infty}(\Omega)^*)} = \sum_{k=1}^{N} \left\| \frac{w_\tau^k - w_\tau^{k-1}}{\tau} \right\|_{W^{1,\infty}(\Omega)^*} = \sum_{k=1}^{N} \sup_{v \in W^{1,\infty}, (\Omega),\, \|v\| \le 1} \int_\Omega \frac{w_\tau^k - w_\tau^{k-1}}{\tau} v \mathrm{d}x$$

$$= \sum_{k=1}^{N} \sup_{v \in W^{1,\infty}(\Omega),\, \|v\| \le 1} \int_\Omega \left(-\mathcal{K}(\lambda_\tau^k, w_\tau^k)\nabla w_\tau^k \cdot \nabla v + \delta_S^*(\dot{\lambda}_\tau(t_k))v + \epsilon|\dot{\lambda}_\tau(t_k)|^q v \right.$$

$$\left. + \Theta(w^k)\vec{a} \cdot \dot{\lambda}_\tau(t_k)v \right) \mathrm{d}x + \int_\Gamma (\bar{b}_\tau \bar{\theta}_{\mathrm{ext},\tau} - \bar{b}_\tau \Theta(w^k))v \mathrm{d}S$$

$$\le \sup_{\tilde{v} \in C([0,T],W^{1,\infty}(\Omega)),\, \|v\| \le 1} \int_Q \left(-\mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau)\nabla \bar{w}_\tau \cdot \nabla \tilde{v} + \delta_S^*(\dot{\lambda}_\tau)\tilde{v} \right.$$

$$\left. + \epsilon|\dot{\lambda}_\tau|^q \tilde{v}\Theta(\bar{w}_\tau)\vec{a} \cdot \dot{\lambda}_\tau \tilde{v} \right) \mathrm{d}x + \int_\Gamma (\bar{b}_\tau \bar{\theta}_{\mathrm{ext},\tau} - \bar{b}_\tau \Theta(\bar{w}_\tau))\tilde{v} \mathrm{d}S \mathrm{d}t.$$

Now because of all the preceding steps we may use the Hölder inequality for all terms on the right-hand side to get estimate (4.11). □

Proposition 4.3 (Convergence and Existence). *Provided* (3.5) *holds, there exist* $(\nu, \lambda, w) \in (\mathcal{Y}^p(\Omega; \mathbb{R}^d))^{[0,T]} \times W^{1,q}([0,T]; L^q(\Omega, \mathbb{R}^{d+1})) \times L^r([0,T], W^{1,r}(\Omega))$ *and a sequence* $\tau \to 0$ *such that*

$$\bar{\lambda}_\tau \overset{*}{\rightharpoonup} \lambda \quad \text{in } L^\infty([0,T]; L^q(\Omega, \mathbb{R}^{d+1})) \text{ and } \bar{\lambda}_\tau(t) \rightharpoonup \lambda(t) \; \forall t \in [0,T] \text{ in } L^q(\Omega; \mathbb{R}^{d+1}), \quad (4.26)$$

$$\dot{\lambda}_\tau \rightharpoonup \dot{\lambda} \quad \text{in } L^q(Q; \mathbb{R}^{d+1}), \quad (4.27)$$

$$\bar{w}_\tau \rightharpoonup w \; L^r([0,T]; W^{1,r}(\Omega)), \; r < \frac{d+2}{d+1} \text{ and } \bar{w}_\tau \to w \text{ in } L^1(Q), \quad (4.28)$$

$$\bar{m}_\tau \rightharpoonup m \quad \text{in } L^2(Q; \mathbb{R}^d), \quad (4.29)$$

$$L \bullet \bar{\nu}_\tau \to L \bullet \nu \quad \text{in } L^2([0,T]; H^{-1}(\Omega; \mathbb{R}^{d+1})). \quad (4.30)$$

Moreover, for each $t \in [0,T]$ *there exists a subsequence* $\tau_{k(t)}$ *such that*

$$\bar{\nu}_{\tau_{k(t)}}(t) \overset{*}{\rightharpoonup} \nu(t) \quad \text{in } L_{\mathrm{w}}^\infty(\Omega; \mathcal{M}(\mathbb{R}^d)). \quad (4.31)$$

Every (ν, λ, w) *obtained in this way is then is a weak solution of* (2.10).

Remark 4.4. Note that the Young measure ν obtained by (4.31) surely *does not need to be measurable as a function of time*. However exploiting the convexity of the magnetic part of the Gibbs free energy and, in particular, its *strict convexity* in the moments $m \equiv \mathrm{id} \bullet \nu$ and $L \bullet \nu$ we shall prove that, in contrast to ν, the moments m and $L \bullet \nu$ are measurable in time.

Proof of Proposition 4.3. Again for lucidity, we divide the proof into six steps.

Step 1: Selection of subsequences: By the a-priori estimates proved in Proposition 4.2 we may find a sequence of τ's and $(\lambda, w) \in (W^{1,q}([0,T]; L^q(\Omega, \mathbb{R}^{d+1})) \times L^r([0,T], W^{1,r}(\Omega))$ such that (4.26)–(4.28) hold. Moreover, there exists $\Xi \in L^2(Q; \mathbb{R}^{d+1})$ and $\tilde{h} \in L^(2[0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that

$$L \bullet \bar{\nu}_\tau \rightharpoonup \Xi \qquad\qquad \text{in } L^2(Q; \mathbb{R}^{d+1}), \quad (4.32)$$

$$\nabla u_{\bar{m}_\tau} \rightharpoonup \tilde{h} \qquad\qquad \text{in } L^2([0,T] \times \mathbb{R}^d; \mathbb{R}^d). \quad (4.33)$$

Indeed, by (4.7) we know that λ_τ is (considering also the integrability of the initial condition, cf. (3.5g)) bounded in $W^{1,q}([0,T]; L^q(\Omega; \mathbb{R}^{d+1}))$ and hence converges weakly to some λ in this space (for the subsequence selected). As $W^{1,q}(I; L^q(\Omega, \mathbb{R}^{d+1})) \subset C([0,T]; L^q(\Omega, \mathbb{R}^{d+1}))$ we also get that $\lambda_\tau(t) \rightharpoonup \lambda(t)$ for all $t \in [0,T]$ in $L^q(\Omega, \mathbb{R}^{d+1})$.

Moreover, $\bar{\lambda}_\tau$ converges weakly in $L^q(Q; \mathbb{R}^{d+1})$ to λ as well, because $\|\lambda_\tau - \bar{\lambda}_\tau\|_{L^q(Q;\mathbb{R}^{d+1})} \le \tau\|\dot{\lambda}\|_{L^q(Q;\mathbb{R}^{d+1})} \to 0$. Further, again by (4.7), $\bar{\lambda}_\tau$ is bounded in $L^\infty([0,T]; L^q(\Omega; \mathbb{R}^{d+1}))$ and

also BV$([0,T]; L^q(\Omega; \mathbb{R}^{d+1}))$. Hence, by making use of a slight modification of Helly's theorem [23, 25] $\bar{\lambda}_\tau(t) \rightharpoonup \lambda(t)$ for all $t \in [0,T]$ in $L^q(\Omega; \mathbb{R}^{d+1})$, too. Note also, that due to the fact that $q \geq 2$ and the compact embedding $L^2(\Omega; \mathbb{R}^{d+1}) \Subset H^{-1}(\Omega; \mathbb{R}^{d+1})$, $\bar{\lambda}_\tau(t) \to \lambda(t)$ strongly in $H^{-1}(\Omega; \mathbb{R}^{d+1})$ for all $t \in [0,T]$.

For this sequence of τ's, $L \bullet \bar{\nu}_\tau$ is bounded in $L^2(Q; \mathbb{R}^{d+1})$ due to (4.6); therefore (4.32) follows just by Banach's selection principle. Similarly, also $\bar{m}_\tau$ is bounded in $L^2(Q; \mathbb{R}^d)$ due to (4.6) and hence, by standard theory for the elliptic equation (4.1b), $\nabla u_{\bar{m}_\tau}$ is bounded in $L^2([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$; this readily gives (4.33). Due to (4.9), $\bar{w}_\tau$ is bounded in $L^r([0,T]; W^{1,r}(\Omega))$, $r < \frac{d+2}{d+1}$ and therefore converges weakly to some w in this space. Having the dual estimate on the time derivative of w_τ (4.11), we exploit the Aubin-Lions-lemma generalized for measure-valued derivatives (see [27]) to get that $\bar{w}_\tau$ converges *even strongly* to w in $L^{(d+2)/(d+1)-\beta}([0,T]; W^{1-\beta,(d+2)/(d+1)-\beta}(\Omega))$ and after interpolation $L^{\frac{d+2}{d}-\beta}(Q)$ for any $\beta > 0$ small, so that the traces converge strongly in $L^{(d+2)/(d+1)-\beta}([0,T]; L^{(d^2+d-2)/(d^2-2)-\beta}(\Gamma))$; for any small $\beta > 0$ cf. [34, Formulae (4.42) and (4.55)]. Moreover the estimate (4.11) assures that $\underline{w}_\tau$, cf. (4.5), converges strongly in $L^{\frac{d+2}{d}-\beta}(Q)$ *to the same limit as* $\bar{w}_\tau$.

Thanks to the growth condition $|\Theta(w)| \leq |w|^{1/\omega}$ and assumption (3.5e) we have that $|\Theta(w)| \leq |w|^{1/\omega} \leq |w|^{1/q'}$. Hence exploiting the continuity of Nemytskii mappings in Lebesgue spaces and using continuity of Θ, we have that $\Theta(\bar{w}_\tau) \to \Theta(w)$ in $L^{q'}(Q)$. Similarly also $\Theta(\underline{w}_\tau) \to \Theta(w)$ in $L^{q'}(Q)$.

Now, take any $t \in [0,T]$ arbitrary but fixed. Then due to the bound (4.6) select a subsequence $\tau_{k(t)}$ such that $\bar{\nu}_{\tau_{k(t)}}(t) \rightharpoonup \nu(t)$ in $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^d))$. As indicated by the index $k = k(t)$, this selection *may depend on time t*. Estimate (4.6) then also assures that ν is a Young measure. Since the growth of L is strictly smaller than p it holds also that $L \bullet \bar{\nu}_{\tau_{k(t)}}(t) \rightharpoonup L \bullet \nu(t)$ in $L^2(\Omega; \mathbb{R}^{d+1})$ and therefore also $L \bullet \bar{\nu}_{\tau_{k(t)}}(t) \to L \bullet \nu(t)$ strongly in $H^{-1}(\Omega; \mathbb{R}^{d+1})$. Similarly $\nabla u_{\text{id} \bullet \bar{\nu}_{\tau_{k(t)}}(t)} \rightharpoonup \nabla u_{\text{id} \bullet \nu(t)}$ in $L^2(\Omega; \mathbb{R}^d)$. Note that this does not necessarily imply that $L \bullet \nu = \Xi$ or that $\nabla u_{\text{id} \bullet \nu} = \tilde{h}$ (where Ξ and $\tilde{h}$ were defined above).

Step 2: Minimization principle: Let t be still fixed. A direct consequence of (4.1a) is the *discrete minimization principle* that reads as

$$\mathfrak{G}(t_{\tau_{k(t)}}, \bar{\nu}_{\tau_{k(t)}}(t), \bar{\lambda}_{\tau_{k(t)}}(t)) \leq \mathfrak{G}(t_{\tau_{k(t)}}, \hat{\nu}, \bar{\lambda}_{\tau_{k(t)}}(t)), \tag{4.34}$$

for any $\hat{\nu} \in \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ with $\mathfrak{G}$ defined in (4.13). Here we denoted $t_{\tau_{k(t)}} = l \cdot \tau_{k(t)}$, where $l = \min_{s \in \mathbb{N}}\{t \leq s\tau_{k(t)}\}$. Applying $\liminf_{\tau_{k(t)} \to 0}$ on both sides and using the definition of the magnetic part of the Gibbs free energy we get that

$$\liminf_{\tau_{k(t)} \to 0} \mathfrak{G}\big(t_{\tau_{k(t)}}, \bar{\nu}_{\tau_{k(t)}}(t), \bar{\lambda}_{\tau_{k(t)}}(t)\big) \leq \lim_{\tau_{k(t)} \to 0} \mathfrak{G}\big(t_{\tau_{k(t)}}, \hat{\nu}, \bar{\lambda}_{\tau_{k(t)}}(t)\big)$$

$$= \lim_{\tau_{k(t)} \to 0} \int_\Omega \phi \bullet \hat{\nu} - h(t_{\tau_{k(t)}}) \cdot \hat{m} \, \mathrm{d}x + \big\|\bar{\lambda}_{\tau_{k(t)}}(t) - L \bullet \hat{\nu}\big\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})} = \mathfrak{G}(t, \hat{\nu}, \lambda(t)),$$

where $\hat{m} \equiv \text{id} \bullet \hat{\nu}$ because of the continuity of h and the strong convergence of $\bar{\lambda}_\tau(t)$ in $H^{-1}(\Omega; \mathbb{R}^{d+1})$. As to the left-hand side, because of the boundedness from below of ϕ, we may estimate

$$\int_\Omega \big(\phi \bullet \nu(t) - h(t) \cdot m(t)\big) \, \mathrm{d}x + K\big\|\lambda(t) - L \bullet \nu(t)\big\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})}$$

$$\leq \liminf_{\tau_{k(t)} \to 0} \int_\Omega \phi \bullet \bar{\nu}_{\tau_{k(t)}}(t) - h(t_{\tau_{k(t)}}) \cdot \bar{m}_{\tau_{k(t)}}(t) \, \mathrm{d}x + K\big\|\bar{\lambda}_{\tau_{k(t)}}(t) - L \bullet \bar{\nu}_{\tau_{k(t)}}(t)\big\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})},$$

which already gives the sought minimization principle.

Step 3: Measurability of $L \bullet \nu$ and $\nabla u_m(\cdot)$, strong convergence of $L \bullet \nu$: Let t remain fixed as in Step 2. Then, the convexity of $\mathfrak{G}$, the convexity of the space of Young measures and the strict

convexity of $\mathfrak{G}$ in the terms $\lambda - L \bullet \nu$ and ∇u_m, used for $m = \mathrm{id} \bullet \nu$ with ν corresponding to some minimizer of (3.1) and $\lambda = \lambda(t)$ already selected in Step 1, ensures that both $\lambda - L \bullet \nu$ and ∇u_m are determined uniquely. Then, since $\lambda(t)$ is fixed, also $L \bullet \nu$ is determined uniquely, although the minimizer ν does not need to be.

In turn it means that $L \bullet \bar{\nu}_\tau(t) \rightharpoonup L \bullet \nu(t)$ in $L^2(\Omega; \mathbb{R}^{d+1})$ and $\nabla u_{\bar{m}_\tau(t)} \rightharpoonup \nabla u_{m)}$ in $L^2(\Omega; \mathbb{R}^d)$ for the sequence of τ's already selected in Step 1. Hence, by usage of the Lebesgue dominated convergence theorem, $L \bullet \nu(t) = \Xi(t)$ and $\tilde{h} = \nabla u_{m(t)}$ for a.a. $t \in [0, T]$. This in particular shows that both $L \bullet \nu$ and $\nabla u_{m(t)}$ are measurable.

Note that the above implies also $\|L \bullet \bar{\nu}_\tau(t) - L \bullet \nu(t)\|_{H^{-1}(\Omega; \mathbb{R}^{d+1})} \to 0$ for all $t \in [0, T]$. Realizing moreover that $\sup_{t \in [0,T]} \|L \bullet \bar{\nu}_\tau(t) - L \bullet \nu(t)\|^2_{H^{-1}(\Omega; \mathbb{R}^{d+1})} \mathrm{d}t$ is bounded by a constant due to (4.6) we may use *Lebesgue dominated convergence theorem* to get (4.30).

Step 4: Convergence of the flow rule: We are now in the position to pass to the limit in the discrete flow rule, i.e. (4.22) but integrated to T instead of t_k. First choose a test function $v \in L^q(Q; \mathbb{R}^{d+1})$ and consider its piecewise constant approximations v_τ such that $v_\tau \to v$ strongly on $L^q(Q; \mathbb{R}^{d+1})$ and moreover $\|v_\tau\|_{L^{2q}(Q; \mathbb{R}^{d+1})} \leq C\tau^{-1/(2q+1)}$. By convexity of $\delta_S^*(\cdot) + \frac{\epsilon}{q}|\cdot|^q$, we get that

$$\int_Q \left(\delta_S^*(\dot{\lambda}) + \frac{\epsilon}{q}|\dot{\lambda}|^q\right)\mathrm{d}x\mathrm{d}t \leq \liminf_{\tau \to 0} \int_Q \left(\delta_S^*(\dot{\lambda}_\tau) + \frac{\epsilon}{q}|\dot{\lambda}_\tau|^q\right)\mathrm{d}x\mathrm{d}t + \int_\Omega \tau|\bar{\lambda}_\tau(T)|^{2q}\mathrm{d}x.$$

As to the convergence right-hand-side of (4.22), we use that $\Theta(\underline{w}_\tau) \to \Theta(w)$ in $L^{q'}(Q)$ to pass to the limit in $\int_Q \Theta(\underline{w}_\tau) - \theta_c)\vec{a} \cdot (v_\tau - \dot{\lambda}_\tau)\mathrm{d}x\mathrm{d}t$ and (4.26) as well as (4.30) to establish the convergence of $\int_0^T \varkappa \langle\!\langle \bar{\lambda}_\tau - L \bullet \bar{\nu}_\tau, v_\tau - \dot{\lambda}_\tau \rangle\!\rangle \mathrm{d}t$. Since $\|\lambda_{0,\tau}\|_{L^{2q}(\Omega; \mathbb{R}^{d+1})} \leq C\tau^{-1/(2q+1)}$ the term $\tau \int_\Omega |\lambda_0|^{2q}\mathrm{d}x$ converges to zero. Similarly $2q\tau|\bar{\lambda}_\tau|^{2q-2}\bar{\lambda}_\tau v_\tau$ can be pushed to zero thanks to (4.8) and the blow-up for v_τ specified above that allow us to estimate $|\int_Q 2q\tau|\bar{\lambda}_\tau|^{2q-2}\bar{\lambda}_\tau v_\tau|\mathrm{d}x\mathrm{d}t \leq 2q\tau\|\bar{\lambda}_\tau\|^{2q-1}_{L^{2q}(Q; \mathbb{R}^{d+1})}\|v_\tau\|_{L^{2q}(Q; \mathbb{R}^{d+1})} \leq C\tau^{\frac{1}{4q^2+2q}}$. Altogether, applying $\liminf_{\tau \to 0}$ to both sides of the discrete flow rule, we get

$$\int_Q \delta_S^*(\dot{\lambda}) + \frac{\epsilon}{q}|\dot{\lambda}|^q\mathrm{d}x\mathrm{d}t \leq \int_0^T \Bigg(\varkappa \langle\!\langle \lambda - L \bullet \nu, v - \dot{\lambda} \rangle\!\rangle$$
$$+ \int_\Omega \left((\Theta(w) - \theta_c)\vec{a}\cdot(v - \dot{\lambda}) + \delta_S^*(v) + \frac{\epsilon}{q}|v|^q\right)\mathrm{d}x \Bigg)\mathrm{d}t, \quad (4.35)$$

for any $v \in L^q(Q; \mathbb{R}^{d+1})$.

Step 5: Strong convergence of $\dot{\lambda}_\tau$: First test the discrete flow rule (cf. (4.22), *reformulated using the convexity of $|\cdot|^q$*) by $\dot{\lambda}_{S,\tau}$ being a piecewise constant approximation of the function $\dot{\lambda}$ such that $\dot{\lambda}_{S,\tau} \to \dot{\lambda}$ strongly in $L^q(Q; \mathbb{R}^{M+1})$ and moreover $\|\dot{\lambda}_{S,\tau}\|_{L^{2q}(Q; \mathbb{R}^{M+1})} \leq C\tau^{-1/(2q+1)}$. We get

$$\int_Q \delta_S^*(\dot{\lambda}_\tau)\mathrm{d}x\mathrm{d}t + \int_\Omega \tau|\bar{\lambda}_\tau(T)|^{2q}\mathrm{d}x \leq \int_\Omega \tau|\lambda_0|^{2q}\mathrm{d}x + \int_0^T \Bigg(\varkappa \langle\!\langle \bar{\lambda}_\tau - L \bullet \bar{\nu}_\tau, \dot{\lambda}_{S,\tau} - \dot{\lambda}_\tau \rangle\!\rangle$$
$$+ \int_\Omega \left(\delta_S^*(\dot{\lambda}_{S,\tau}) + \epsilon|\dot{\lambda}_\tau|^{q-2}\dot{\lambda}_\tau\cdot(\dot{\lambda}_{S,\tau} - \dot{\lambda}_\tau) + (\Theta(\underline{w}_\tau) - \theta_c)\vec{a}\cdot(\dot{\lambda}_{S,\tau} - \dot{\lambda}_\tau) + 2q\tau|\bar{\lambda}_\tau|^{2q-2}\bar{\lambda}_\tau\cdot\dot{\lambda}_{S,\tau}\right)\mathrm{d}x \Bigg)\mathrm{d}t.$$
$$(4.36)$$

Symmetrically we test the continuous flow rule reformulated as above by $\dot{\lambda}_\tau$ to get

$$\int_Q \delta_S^*(\dot{\lambda})\mathrm{d}x\mathrm{d}t \le \int_0^T \Bigg(\varkappa \langle\!\langle \lambda - L \bullet \nu, \dot{\lambda}_\tau - \dot{\lambda} \rangle\!\rangle$$
$$+ \int_\Omega \Big(\epsilon|\dot{\lambda}|^{q-2}\dot{\lambda}\cdot(\dot{\lambda}_\tau - \dot{\lambda}) + (\Theta(\bar{w}) - \theta_c)\vec{a}\cdot(\dot{\lambda}_\tau - \dot{\lambda}) + \delta_S^*(v) \Big)\mathrm{d}x \Bigg)\mathrm{d}t. \quad (4.37)$$

We add (4.36) and (4.37), apply Hölder inequality and $\lim_{\tau\to 0}$ to estimate

$$\epsilon \lim_{\tau\to 0} \Big(\|\dot{\lambda}_\tau\|_{L^q(Q;\mathbb{R}^{d+1})}^{q-1} - \|\dot{\lambda}\|_{L^q(Q;\mathbb{R}^{d+1})}^{q-1} \Big) \Big(\|\dot{\lambda}_\tau\|_{L^q(Q;\mathbb{R}^{d+1})} - \|\dot{\lambda}\|_{L^q(Q;\mathbb{R}^{d+1})} \Big)$$

$$\le \lim_{\tau\to 0} \epsilon \int_Q \Big(|\dot{\lambda}_\tau|^{q-2}\dot{\lambda}_\tau - |\dot{\lambda}|^{q-2}\dot{\lambda} \Big) \cdot (\dot{\lambda}_\tau - \dot{\lambda})\mathrm{d}x\mathrm{d}t$$

$$\le \lim_{\tau\to 0} \Bigg(\int_Q \Big(\underbrace{\epsilon|\dot{\lambda}_\tau|^{q-2}\dot{\lambda}_\tau(\dot{\lambda}_{S,\tau} - \dot{\lambda}) + \delta_S^*(\dot{\lambda}_{S,\tau}) - \delta_S^*(\dot{\lambda})}_{(I)} \Big)\mathrm{d}x\mathrm{d}t$$

$$+ \int_Q \Big(\underbrace{(\Theta(\underline{w}_\tau) - \theta_c)\vec{a}\cdot(\dot{\lambda}_{S,\tau} - \dot{\lambda}_\tau) + \Theta(w)\vec{a}\cdot(\dot{\lambda}_\tau - \dot{\lambda})}_{(II)} \Big)\mathrm{d}x\mathrm{d}t + \int_0^T \underbrace{\varkappa\langle\!\langle \lambda - L\bullet\nu, \dot{\lambda}_\tau - \dot{\lambda}\rangle\!\rangle}_{(III)}\mathrm{d}t$$

$$+ \int_0^T \Big(\underbrace{\varkappa\langle\!\langle \lambda_\tau - L\bullet\bar{\nu}_\tau, \dot{\lambda}_{S,\tau} - \dot{\lambda}_\tau\rangle\!\rangle}_{(IV)} + \int_\Omega \underbrace{2q\tau|\bar{\lambda}_\tau|^{2q-2}\bar{\lambda}_\tau\dot{\lambda}_{S,\tau}}_{(V)}\mathrm{d}x \Big)\mathrm{d}t + \int_\Omega \underbrace{\tau|\lambda_0|^{2q}}_{(VI)} \Bigg) \le 0.$$

When passing to the limit on the right-hand-side use that $\dot{\lambda}_{S,\tau} \to \dot{\lambda}$ in $L^q(Q;\mathbb{R}^{d+1})$ to limit Term (I) to 0. For Term (II) we use the convergences established in Step 1, and the convergence of $\Theta(\underline{w}_\tau) \to \Theta(w)$ in $L^{q'}(Q;\mathbb{R}^{d+1})$ to see its limit being 0. When turning to Term (III), (4.27) needs to be applied to find this term approaching again 0. Term (IV) converges to 0 by applying (4.26) and (4.30) combined with $\dot{\lambda}_{S,\tau} \to \dot{\lambda}$ in $L^q(Q;\mathbb{R}^{d+1})$. Term (V) can be pushed to 0 similarly as when converging the flow rule using the available blow-up conditions, Term (VI) converges also to 0 by exploiting (4.2a). Passing then to the limit and using all above said, we arrive at $\|\dot{\lambda}_\tau\|_{L^q(Q;\mathbb{R}^{d+1})} \to \|\dot{\lambda}\|_{L^q(Q;\mathbb{R}^{d+1})}$. Hence, by the local convexity of $L^q(Q;\mathbb{R}^{d+1})$, the already proved weak convergence $\dot{\lambda}_\tau \to \dot{\lambda}$ in $L^q(Q;\mathbb{R}^{d+1})$ turns to be strong.

Step 6: Convergence of the enthalpy equation: Performing discrete by parts integration in (4.1c)) yields

$$-\int_Q \big(\bar{w}_\tau\dot{\varphi} + \mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau)\nabla\bar{w}_\tau\nabla\bar{\varphi} \big)\mathrm{d}x + \int_\Gamma \bar{b}_\tau\Theta(\bar{w}_\tau)\bar{\varphi}\mathrm{d}S\mathrm{d}t = \int_\Omega w_{0,\tau}\bar{\varphi}(0)\mathrm{d}x$$
$$+ \int_Q \big(\delta_S^*(\dot{\lambda}_\tau)\bar{\varphi} + \epsilon|\dot{\lambda}_\tau|^q\bar{\varphi} \big)\mathrm{d}x\mathrm{d}t + \int_Q \Theta(\bar{w}_\tau)\vec{a}\cdot\dot{\lambda}_\tau\bar{\varphi}\mathrm{d}x + \int_\Gamma \bar{b}_\tau\bar{\theta}_{\mathrm{ext},\tau}\bar{\varphi}\mathrm{d}S\mathrm{d}t, \quad (4.38)$$

for all $\bar{\varphi}$ piecewise constant on the intervals $(t_{j-1}, t_j]$ such that $\bar{\varphi}(t_j, \cdot) \in H^1(\Omega)$, $\bar{\varphi}(T) = 0$ and φ piecewise linear on the intervals $(t_{j-1}, t_j]$, such that $\bar{\varphi}(t_j, \cdot) = \varphi(t_j, \cdot)$. Note that by such test functions we may approximate (strongly in the norm of $L^p(Q)$, $p \in [1, \infty]$) any $\tilde{\varphi} \in C^1(\bar{Q})$.

To make a limit passage for $\tau \to 0$ in this equation, we make use of (4.28) (and the approximation of φ mentioned above) to handle the term $\bar{w}_\tau\dot{\varphi}$. Then use that $\bar{\lambda}_\tau \to \lambda$ strongly in $L^q(Q;\mathbb{R}^{d+1})$, which together with (3.5e) gives $\mathcal{K}(\bar{\lambda}_\tau, \bar{w}_\tau) \to \mathcal{K}(\lambda, w)$ strongly in any Lebesgue space, except for $L^\infty(Q;\mathbb{R}^{d\times d})$. Exploit also that $\bar{w}_\tau \to w$ strongly in $L^{(d+2)/(d+1)-\beta}([0,T]; W^{1-\beta,(d+2)/(d+1)-\beta}(\Omega))$ for any $\beta > 0$ small (as shown in Step 1), so that the traces converge strongly in $L^{(d+2)/(d+1)-\beta}([0,T]; L^{(d^2+d-2)/(d^2-2)-\beta}(\Gamma))$. Combining that with (3.6) allows us to handle the left-hand-side boundary term. For the right-hand-side we exploit the strong convergence $\dot{\lambda}_\tau \to \dot{\lambda}$ in $L^q(Q;\mathbb{R}^{d+1})$ to the limit in the terms expressing

dissipated heat. For the term $\Theta(\bar{w}_\tau)\vec{a}\cdot\dot{\bar{\lambda}}_\tau\bar{\varphi}$ exploit $\Theta(\bar{w}_\tau) \to \Theta(w)$ in $L^{q'}(Q)$, for the right-hand-side boundary term we have weak convergence of $\bar{b}_\tau\bar{\theta}_{\mathrm{ext},\tau} \rightharpoonup b\theta_{\mathrm{ext}}$ in $L^1(\Sigma)$, which is enough to establish the limit of this term. $\qquad\square$

5. Less dissipative modification of the model

In some cases it might be advantageous to generalize the presented model in the following way: assume that only one part of the internal parameter λ, which we denote $A\lambda$, to be subjected to rate-dependent dissipation as before while the other part $\lambda - A\lambda$ evolves purely rate-independently; here $A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is a linear *projection*. Assume moreover that the coupling between the magnetic and thermic part is realized *only through the rate-dependent part of the vector of volume fractions*, i.e. $\mathrm{Ker}\,A \subset \mathrm{Ker}(\vec{a}\cdot)$. We alter naturally the dissipation potential

$$R(\dot{\lambda}) := \int_\Omega \delta_S^*(\dot{\lambda}) + \frac{\epsilon}{q}|A\dot{\lambda}|^q \,\mathrm{d}x. \tag{5.1}$$

We further suppose that the rate-independent part can be split as $\delta_S^*(\dot{\lambda}) = \delta_{S_2}^*(A\dot{\lambda}) + \delta_{S_1}^*(A^\perp\dot{\lambda})$ with $A^\perp\dot{\lambda} := \dot{\lambda} - A\dot{\lambda}$, and leave the Gibbs free energy unchanged as in (2.4). The evolution of the system is then again governed by (2.10). Now, however, the system hosts a rate-independent process in evolution of the component $A^\perp\lambda$. In the spirit of [34] we can formulate this system weakly by exploiting the concept of *semi-stability* combined with energy balance:

Definition 5.1. *We call the triple* $(\nu, \lambda, w) \in (\mathscr{Y}^p(\Omega;\mathbb{R}^d))^{[0,T]} \times \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d+1})) \times L^1([0,T];W^{1,1}(\Omega))$ *such that* $m \equiv \mathrm{id}\bullet\nu \in L^2(Q;\mathbb{R}^d)$, $L\bullet\nu \in L^2(Q;\mathbb{R}^{d+1})$ *and* $A\dot{\lambda} \in L^q(\Omega;\mathbb{R}^{d+1})$ *a* weak solution *to* (2.13) *with* $\epsilon|\dot{\lambda}|^{q-2}\dot{\lambda}$ *in* (2.13b) *and* $\epsilon|\dot{\lambda}|^q$ *in* (2.13c) *replaced respectively by* $\epsilon|A\dot{\lambda}|^{q-2}A\dot{\lambda}$ *and* $\epsilon|A\dot{\lambda}|^q$ *if it satisfies:*

1. *The* **semistability***: For all* $\tilde{\nu}$ *in* $\mathscr{Y}^p(\Omega;\mathbb{R}^d)$, *all* $\tilde{\lambda} \in H^{-1}(\Omega;\mathbb{R}^{d+1})$ *such that* $A\tilde{\lambda} = 0$ *and all* $t \in [0,T]$

$$\mathscr{G}(t,\nu,A^\perp\lambda,\Theta(w)) \le \mathscr{G}(t,\tilde{\nu},\tilde{\lambda},\Theta(w)) + \int_\Omega \delta_{S_1}^*(A^\perp(\tilde{\lambda}-\lambda))\,\mathrm{d}x. \tag{5.2}$$

2. *The* **rate dependent flow rule***:*(3.3) *with* $A\dot{\lambda}$ *instead of* $\dot{\lambda}$ *and* Av *instead of* v.
3. *The* **total energy balance**

$$\mathfrak{G}(T,\nu(T),\lambda(T)) + \int_\Sigma b\Theta(w)\,\mathrm{d}S\mathrm{d}t \le \mathfrak{G}(0,\nu_0,\lambda_0) + \int_\Omega w_0\mathrm{d}x$$

$$+ \int_\Sigma \theta_{\mathrm{ext}}\,\mathrm{d}S\mathrm{d}t + \int_0^T \mathfrak{G}'_t(t,\nu_u,\bar{\lambda}_\tau)\,\mathrm{d}t. \tag{5.3}$$

4. *The* **enthalpy equation***:* (3.4) *with* $\int_{\bar{Q}} \varphi\,\mathcal{H}(\mathrm{d}x\mathrm{d}t) + \int_Q \delta_{S_2}^*(A\dot{\lambda})\varphi\mathrm{d}x\mathrm{d}t$ *instead of* $\int_Q \delta_S^*(\dot{\lambda})\varphi\mathrm{d}x\mathrm{d}t$ *and* $\int_Q \epsilon|A\dot{\lambda}|^q\varphi\mathrm{d}x\mathrm{d}t$ *instead of* $\int_Q \epsilon|\dot{\lambda}|^q\varphi\mathrm{d}x\mathrm{d}t$; *here we denoted* $\mathcal{H} \in \mathcal{M}(\bar{Q})$ *the measure (=heat production rate by rate-independent dissipation) defined by prescribing its values on every closed set* $A = [t_1,t_2] \times B$, *where* $B \subset \Omega$ *is a Borel set as*

$$\mathcal{H}(A) = \mathrm{Var}_{\delta_{S_1}^*}(\lambda|_B; t_1, t_2).$$

5. **remaining initial conditions** *and the* **reduced Maxwell system** (3.2).

Theorem 5.2. *Let* (3.5) *hold and* A *be a projector* $\mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$. *Then at least one weak solution to* (2.10) *in the enthalpy formulation in accord with Definition 5.1 does exist and also* (3.7) *holds but weakened as* $\frac{\partial w}{\partial t} \in L^1(I;W^{1,\infty}(\Omega)^*) + \mathcal{M}(\bar{Q})$.

Sketch of the proof. We proceed in a similar way as above by discretizing the problem in time and introducing a time-incremental problem (4.1a) with the discrete version of $|\dot{\lambda}|^q$ replaced by $|A\dot{\lambda}|^q$.

The existence of approximate solutions and also a-priori estimates are proved similarly as above with obvious modifications where there are needed; in particular Step 1 of Proposition 4.2 gives the boundedness of $\bar{\lambda}_\tau$ in $L^\infty([0,T]; H^{-1}(\Omega; \mathbb{R}^{d+1})) \cap \mathrm{BV}([0,T]; L^1(\Omega; \mathbb{R}^{d+1}))$.

As to convergence, even in this case, we can establish obvious modifications of convergences (4.26)-(4.31) (we replace $\dot{\lambda}$ by $A\dot{\lambda}$ when necessary). To obtain the semistability, we first note that a direct consequence of the time-incremental problem is the *discrete semi-stability condition*

$$\mathfrak{G}(t_{\tau_{k(t)}}, \bar{\nu}_{\tau_{k(t)}}(t), A^\perp \lambda_{\tau_{k(t)}}(t)) \le \mathfrak{G}(t_{\tau_{k(t)}}, \hat{\nu}, \hat{\lambda}) + \int_\Omega \delta_S^*(A^\perp(\hat{\lambda} - \lambda_{\tau_{k(t)}}))\mathrm{d}x, \qquad (5.4)$$

for any $\hat{\nu} \in \mathscr{Y}^p(\Omega; \mathbb{R}^d)$ and any $\hat{\lambda} \in H^{-1}(\Omega; \mathbb{R}^{d+1})$ such that $A\hat{\lambda} = 0$. This semistability can be converged similarly as above, however as we do not know that $\lambda_{\tau_{k(t)}}(t) \to \lambda(t)$ strongly in $L^1(\Omega; \mathbb{R}^{d+1})$ (would be necessary to establish the convergence of the right-hand-side term $\delta_S^*(A^\perp(\hat{\lambda} - \lambda_{\tau_{k(t)}}))$ we use that $\mathfrak{G}$ is *quadratic* in λ and employ the so-called *binomic trick* as in e.g. [34]. This trick is based on choosing the test function as $\hat{\lambda} = \tilde{\lambda} - \lambda(t) + \lambda_{\tau_{k(t)}}(t)$ (where $\tilde{\lambda}$ is arbitrary but such that $A\tilde{\lambda} = 0$), subtracting the right from the left hand side of the semistability, use the binomic formula and then passing to the limit.

The flow rule can be converged in the same manner as above, also we obtain the strong convergence of $A\dot{\lambda}_\tau \to A\dot{\lambda}$ in $L^q(Q; \mathbb{R}^{d+1})$. The only delicate part in this case is the convergence of the heat equation, in particular the convergence of the right-hand-side rate-independent dissipation terms. To be able to find the limit of this term, it is necessary to prove the discrete energy inequality

$$\mathfrak{G}(T, \bar{\nu}_\tau(T), \bar{\lambda}_\tau(T)) + \int_\Omega \tau|\bar{\lambda}_\tau(T)|^{2q}\mathrm{d}x + \int_Q \left(\delta_S^*(\dot{\lambda}_\tau) + \epsilon|A\dot{\lambda}_\tau|^q\right)\mathrm{d}x\mathrm{d}t$$

$$\le \mathfrak{G}(0, \bar{\nu}_\tau(0), \bar{\lambda}_\tau(0)) + \int_\Omega \tau|\lambda_{\tau,0}|^{2q}\mathrm{d}x + \int_0^T \mathfrak{G}_t'(t, \underline{\nu}_\tau, \underline{\lambda}_\tau). \qquad (5.5)$$

To get this inequality, we exploit the *convexity of the problem* and realize that solutions of the original time-incremental problem are also solutions of the following auxiliary minimization problem

$$\begin{aligned}
\text{Minimize} \quad &\mathscr{G}(t_k, \nu, \lambda, \Theta(w_\tau^{k-1})) + \int_\Omega \left(\tau|\lambda|^{2q} + \tau\delta_S^*\left(\frac{\lambda - \lambda_\tau^{k-1}}{\tau}\right)\right. \\
&\left. +\tau\epsilon\left|\frac{A\lambda_\tau^k - A\lambda_\tau^{k-1}}{\tau}\right|^{q-2}\left(\frac{A\lambda_\tau^k - A\lambda_\tau^{k-1}}{\tau}\right)\left(\frac{A\lambda - \lambda\tau^{k-1}}{\tau}\right)\right)\mathrm{d}x \\
\text{subject to} \quad &(\nu, \lambda) \in \mathscr{Y}^p(\Omega; \mathbb{R}^d) \times L^{2q}(\Omega; \mathbb{R}^{d+1})
\end{aligned} \right\} \qquad (5.6)$$

and vice versa. Inequality (5.5) is then got by testing (5.6) by $(\nu_\tau^{k-1}, \lambda_\tau^{k-1})$. Using (5.5) and also the "inverse limit energy inequality"

$$\mathfrak{G}(T, \nu(T), \lambda(T)) + \mathrm{Var}_{\delta_{S_1}^*}(A^\perp \lambda; [0,T]) + \int_Q \delta_{S_2}^*(A\dot{\lambda}) + \epsilon|A\dot{\lambda}|^q\mathrm{d}x\mathrm{d}t$$

$$\ge \mathfrak{G}(0, \nu(0), \lambda(0)) + \int_0^T \mathfrak{G}_t'(t, \nu(t), \lambda(t))\mathrm{d}t \qquad (5.7)$$

that is a consequence of the semistability (see e.g. [34]) we can estimate

$$\int_{\bar{Q}} \mathcal{H}(\mathrm{d}x\mathrm{d}t) = \mathrm{Var}_{\delta_{S_1}^*}(A^\perp\lambda, [0,T]) \leq \liminf_{\tau\to 0} \int_Q \delta_{S_1}^*(\dot{\lambda}_\tau)\mathrm{d}x\mathrm{d}t \leq \limsup_{\tau\to 0} \int_Q \delta_{S_1}^*(\dot{\lambda}_\tau)\mathrm{d}x\mathrm{d}t$$

$$\leq \limsup_{\tau\to\infty} \int_0^T \left(\mathfrak{G}_t'(t, \bar{\nu}_\tau(t)) - \int_\Omega \left(\delta_{S_2}^*(A\dot{\lambda}_\tau) + \epsilon|A\dot{\lambda}_\tau|^q \right)\mathrm{d}x \right)\mathrm{d}t - \mathfrak{G}(T, \bar{\nu}_\tau(T), \bar{\lambda}_\tau(T))$$

$$+ \mathfrak{G}(0, \nu_0, \lambda_0) - \int_\Omega \left(\tau|\bar{\lambda}_\tau(T)|^{2q} + \tau|\bar{\lambda}_\tau(0)|^{2q} \right)\mathrm{d}x$$

$$\leq \int_0^T \left(\mathfrak{G}_t'(t, \nu(t)) - \int_\Omega \left(\delta_{S_2}^*(A\dot{\lambda}) + \epsilon|A\dot{\lambda}|^q \right)\mathrm{d}x \right)\mathrm{d}t - \mathfrak{G}(T, \nu(T), \lambda(T)) + \mathfrak{G}(0, \nu_0, \lambda_0)$$

$$\leq \mathrm{Var}_{\delta_{S_1}^*}(A^\perp\lambda, [0,T]),$$

which yields $\lim_{\tau\to 0} \int_Q \delta_{S_1}^*(\dot{\lambda}_\tau)\mathrm{d}x\mathrm{d}t = \mathrm{Var}_{\delta_{S_1}^*}(A^\perp\lambda, [0,T])$ and hence allows us to perform the limit passage in the heat equation similarly as above. Note that, due to the form of $\mathfrak{G}$, $\mathfrak{G}_t'$ does not depend on λ. $\qquad\square$

Acknowledgments

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